in *Imaging Technology and Telescopes*, **Proc. SPIE 4091A-06**, 30 July 2000, San Diego, CA

MTF and Integration Time versus Fill Factor for Sparse-Aperture Imaging Systems

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ABSTRACT:

Telescopes and imaging interferometers with sparsely filled apertures can be lighter weight and less expensive than conventional filled-aperture telescopes. Sparse-aperture systems can be characterized by their fill factor, which is the ratio of the area of a given aperture to the area of a filled aperture having comparable theoretical resolution. We show that the modulation transfer function (MTF) at the midrange spatial frequencies tends to be proportional to the fill factor. Using signal-to-noise ratio (SNR) expressions for various sources of noise, we derive the relationship between the integration time, needed to achieve a given SNR, and the fill factor. For example, for a fixed, sparse array, the integration time is proportional to the inverse cube of the fill factor for photon noise, to the inverse square of the fill factor for readout noise, and to the inverse fourth-power of the fill factor for dark current.

Keywords: Telescopes, sparse apertures, imaging, signal-to-noise ratio, image reconstruction

1. BACKGROUND

Since the diffraction-limited resolution of an object viewed through a telescope is inversely proportional to the diameter of the telescope, there is a drive to make telescopes as large as possible. For example, the Next Generation Space Telescope (NGST — http://ngst.gsfc.nasa.gov/) is planned to have a diameter of about 8 m. To reduce cost and make the telescope light weight, it can be advantageous to make the primary mirror of the telescope only partially filled, or sparse. Unfortunately, a sparse aperture causes a substantial depression in the modulation transfer function (MTF) of the system, resulting in a low-contrast and fuzzy raw image. This can be compensated by image reconstuction, such as a Wiener-Helstrom filter, but at the expense of greater noise sensitivity than for a filled aperture. Figure 1 shows an example. When no noise was added, the reconstructed image (not shown) looked like the original, but when there were 1,000 photons per pixel, then the reconstructed image, shown in Figure 1(f), while superior to the raw image, is substantially degraded from the original.

An important question is: as the aperture becomes more sparse, how quickly does it degrade image quality? Since the signal-to-noise ratio (SNR) can be increased by longer integration (exposure) times, it makes sense to ask a related question: to maintain a given desired image quality as the aperture becomes more sparse, by what factor must one increase the integration time?[1] In this paper we analyze the relationship between SNR, MTF, integration time, and sparse-aperture fill factor, and give scaling laws for sparse apertures.

2. SIGNAL-TO-NOISE RATIO

We model the image formation process as

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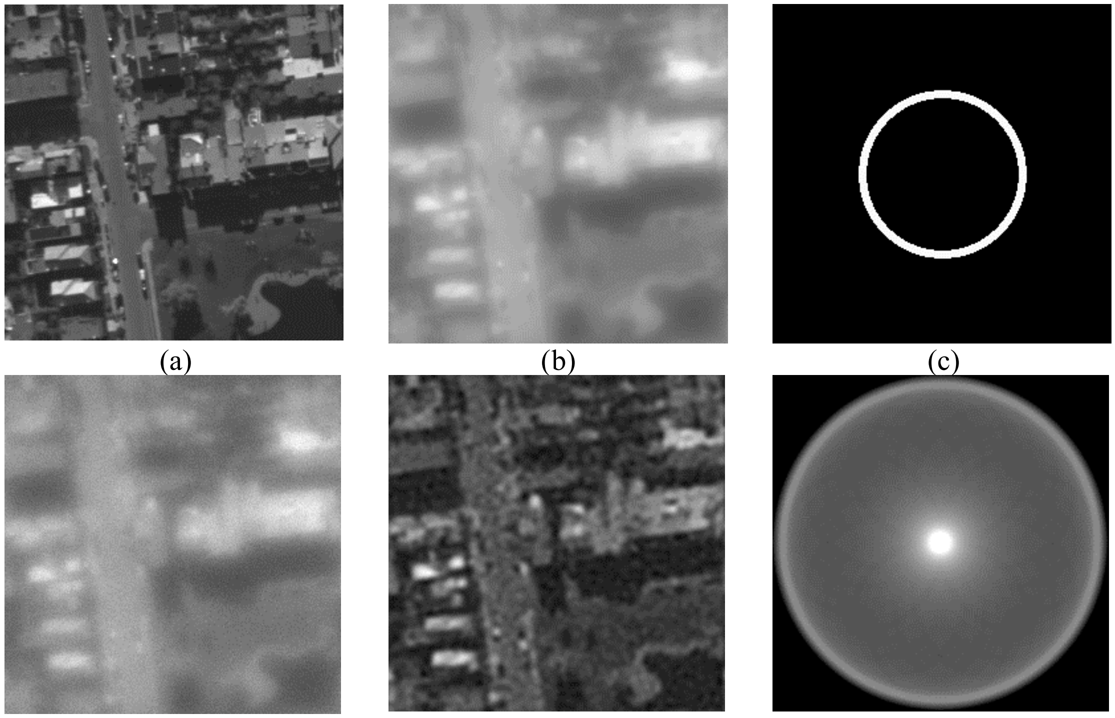

Figure 1. Example of imaging through sparse aperture. (a) Original aerial photograph, (b) diffraction-limited image though annular aperture, (c) annular aperture, (d) raw image [photon noise applied to (b)], (e) Wiener-Helstrom-filtered version of (d), (f) MTF of annular aperture.

$$\begin{aligned} g(x,y) &= Poisson[f(x,y)*s(x,y)+n_b(x,y)+n_{dc}(x,y)]+\sigma_r n_4(x,y)-n_b-n_{dc} \\ &\approx f(x,y)*s(x,y)+\sqrt{f(x,y)*s(x,y)}\,n_1(x,y)+\sqrt{n_b}\,n_2(x,y)+\sqrt{n_{dc}}\,n_3(x,y)+\sigma_r n_4(x,y) \end{aligned} \qquad (1)$$

where $f =$ object brightness (in photons), $s = PSF, \sigma_r =$ rms readout noise

$n_b =$ additive bias (path radiance, haze), $n_{dc} =$ dark current,

$n_k(x,y) =$ independent zero - mean Gaussian noises (unit variance).

In the Fourier domain this becomes

$$G(u,v) = F(u,v)S(u,v) + FT[\sqrt{f(x,y)*s(x,y)}]*N_1(u,v)+\sqrt{n_b}N_2(u,v)+\sqrt{n_{dc}}N_3(u,v)+\sigma_r N_4(u,v) \qquad (2)$$

where the symbols in upper case are the Fourier transforms of the corresponding symbols in lower case. The frist term in Eq. (2) is the signal and the remaining terms are noise. It can be shown that for extended scenes the first noise term is uniformly distributed in the Fourier domain, as are the other noise terms. From this expression we see that the SNR of the Fourier amplitude data is

$$SNR_1(u,v) = \frac{(P/M)S(u,v)\mu(u,v)}{\left[\left(P/M^2\right)+n_b+n_{dc}+\sigma_r^2\right]^{1/2}} \qquad (3)$$

where

$P =$ total number of photons in MxM image, and

$\mu(u,v) = F(u,v)/F(0,0) =$ normalized scene Fourier transform.

This expression, when reduced to the case of only photon noise, is in agreement with Ref. 2. In this expression, the noise term in the denominator is independent of spatial frequency, (u, v), but the signal term in the numerator generally tends to drop quickly with increasing spatial frequency. One can similarly compute the SNR for the squared amplitude, the power spectrum; although one gets a different SNR expression, the conclusions of this paper on scaling laws remain the same. The dependence of μ on spatial frequency is an important factor with regard to image quality, but it does not affect our conclusions on scaling laws. To arrive at the scaling laws, we need the dependencies on the fill factor, a, of the telescope, which we (somewhat arbitrarily) define as the ratio of the area of the sparse aperture to the area of an equivalent-diameter filled aperture that would give the same resolution if there were no noise. If we let

Φ_o = image photons per unit aperture - area per unit time

a = fill factor

A_f = area of filled aperture giving same resolution

η_t = total efficiency (system throughput)x(detector quantum efficiency)

T = integration time

then the total number of detected photons is given by

$$P = \Phi_o a A_f \eta_t T \quad . \tag{4}$$

The MTF also depends on the fill factor. As shown in Figure 2, for the case of the annular aperture, the MTF decreases rapidly at lower spatial frequencies, experiences a large plateau area over which the MTF is relatively constant, and then (after a short blip up) decreases to zero at the diffraction cut-off. This analysis ignores aberrations. We have found that many other well-designed sparse apertures (e.g., Golay apertures having large subapertures) share the same primary feature of a broad plateau region. We have also found that the value of the MTF in this middle-frequency plateau region is typically proportional to the fill factor:

$$S(u_{mid}, v_{mid}) \approx \eta_A a \tag{5}$$

where η_A is a proportionality constant that depends on the aperture type.

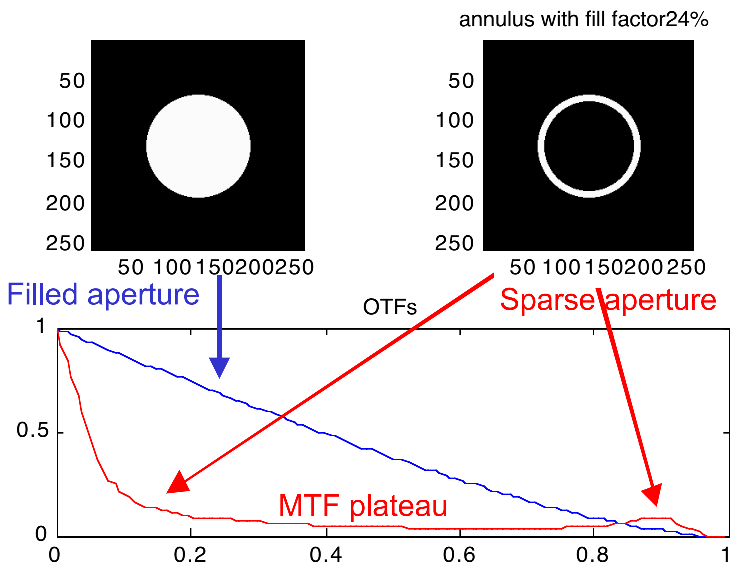

Figure 2. A filled circular aperture, a sparse annular aperture, and their MTFs.

With these factors in hand and using the relations

$M^2 n_b = \Phi_b a A_f \eta_t T$

$n_d = r_{dc} T$ [(dark current rate) x T]

we have

$$SNR_1(u_{mid}, v_{mid}) = \frac{\left(\Phi_o A_f \eta_t \eta_A a^2 T / M\right) \mu(u,v)}{\left[\left(\Phi_o + \Phi_b\right) A_f \eta_t \, aT / M^2 + r_{dc} T + \sigma_r^2\right]^{1/2}} \quad (6)$$

which relates the SNR to the fill factor, integration time, and noise levels.

3. SCALING LAWS

From the SNR expression above we can determine the relationship between the integration time and the fill factor using the following logic. Linear Fourier filtering can boost the MTF to correct for the MTF effect of the sparse aperture, but it does not change the SNR at a given spatial frequency. To achieve a given desired image quality, the SNR at a given spatial frequency must have some minimum threshold value, SNR_t. Fixing all the other factors, we can determine the relationship between a and T by setting the left-hand side of the equation above to SNR_t, and solve for T in terms of a and the other parameters. This gives a messy result (which is omitted here), so for the sake of clarity we do this instead for each noise source individually, setting other noise sources to zero. Then we get the following results.

For the case of photon noise only, we get

$$T = \frac{SNR_t^2}{A_f \eta_t a^3 \left[\eta_A \mu(u,v)\right]^2} \frac{\Phi_o + \Phi_b}{\Phi_o^2}, \quad (7)$$

From this we see that the required integration time is proportional to the inverse cube of the fill factor. This is also immediately evident from the SNR expression for photon noise alone, for which $SNR \sim (a^3 T)^{1/2}$. For example, if at a fill factor of 10% we have just enough light at a given exposure time to get an image of acceptable quality, then if we reduce the fill factor to 1%, a factor of 10 less, the exposure time must increase by a factor of 1,000 in order to get an image of the same quality. This scaling law — this strong dependency of exposure time on fill factor — should be a sobering fact to designers of sparse-aperture telescopes. Nevertheless, for certain cases, such as imaging bright binary star pairs, for which μ drops off slowly with increasing spatial frequency, and one can conveniently use long exposure times, a sparse aperture having a very small fill factor can be quite adequate. At the other extreme, when trying to quickly imaging large, low-contrast, dim objects or scenes, one would be forced to design the fill factor to remain high. Later we summarize the results for the other sources of noise.

The fill factor hurts in two ways: (i) fewer photons arrive at the detector as the aperture loses area, and (ii) the MTF is depressed as the fill factor decreases. The fill-factor cube law arises because the terms in the SNR having to do with the photon signal and its noise are proportional to $a^{1/2} T^{1/2}$, whereas the MTF term is proportional to $a = a^{2/2}$. That is, of the three powers of $a^{1/2}$ balancing against the $T^{1/2}$ term, two of them are from the MTF. Consequently, the depression of the MTF is a bigger factor in the scaling law than the reduction in the number of photons.

So far we have analyzed the case of fixed sparse apertures. Another possibility is to image with telescopes or interferometers that measure only part of spatial frequency space at a given time and synthesize the total frequency-space coverage (the "*u-v* plane") in time as the subapertures move relative to one another.[3,4] An example of this would be imaging with NASA's two-aperture Space Interferometer Mission (SIM[5]). One idea is to rotate the interferometer pair while slowly increasing the distance between the two apertures. In such a case the expression for the SNR in terms of the fill factor and the total integration time, T, is

$$SNR_{synth}(u,v) = \frac{(1/8M)\Phi_o A_f \eta_t a^2 T \mu(u,v)}{\left[(1/4\,M^2)(\Phi_o + \Phi_b) A_f \eta_t a^2 T + (a/4) r_{dc} T + \sigma_r^2\right]^{1/2}} \quad . \tag{8}$$

For the case of photon noise only, we see that $SNR \sim (a^2T)^{1/2}$. Then the total integration time is proportional to only the inverse square of the fill factor, which is an advantage over fixed sparse apertures when the fill factor is low. Table 1 summarizes the scaling laws for fixed and synthetic sparse apertures.

Table 1. Integration time dependency on fill factor, a, for sparse-aperture telescopes and imaging interferometers, for various noise sources.

	Photon & Bias Noise	Read Noise	Dark Current Noise
Fixed Aperture	a^{-3}	a^{-2}	a^{-4}
Synthetic Aperture	a^{-2}	a^{-2}	a^{-3}

The improved performance of synthetic apertures over fixed apertures for all but read noise is due to the noise filtering that they employ. Assuming a synthetic aperture with focal-plane detection, at any one instant the noise is spread uniformly over spatial frequency space, but the signal can be concentrated in a small region. By zeroing out the Fourier domain everywhere except where there is signal, the noise is decreased while the signal is not, giving an increased SNR as compared with a fixed aperture. This advantage is lost for the case of read noise because with a synthetic aperture one must read out the detectors many times as the aperture is synthesized, suffering the read noise each time, whereas for a fixed aperture it is necessary to read out the detectors only once. When read noise dominates, both types of apertures have integration times that are proportional to the inverse square of the fill factor.

5. SUMMARY AND CONCLUSIONS

Sparse apertures can make large-diameter telescopes affordable, but at the price of increased integration time. For example, for a fixed sparse aperture, the integration time to achieve a given image quality goes as the inverse cube of the fill factor. Other scaling laws hold for other noise sources and for synthetic apertures. These scaling laws are important for designing imaging systems with sparse apertures.

ACKNOWLEDGEMENTS

Thanks go to Jet Propulsion Laboratory for providing the original image shown in Figure 1.

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