Improved bounds on object support from autocorrelation support and application to phase retrieval

T. R. Crimmins, J. R. Fienup, and B. J. Thelen

Environmental Research Institute of Michigan, P.O. Box 8618, Ann Arbor, Michigan 48107

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New methods are described for determining tighter upper bounds on the support of an object, given the support of its autocorrelation. These upper bounds are shown, in a digital experiment, to be useful as object-support constraints used with the iterative transform algorithm for solving the phase-retrieval problem.

1. INTRODUCTION

The phase-retrieval problem is the reconstruction of an object $f(x)$ from the modulus $|F(u)|$ of its Fourier transform,

$$F(u) = |F(u)|\exp[i\psi(u)] = \mathcal{F}[f] = \int f(x)\exp(-i2\pi ux)\mathrm{d}x, \qquad (1)$$

where x and u may be one-, two-, or three-dimensional coordinates and $f(x)$ may be complex valued or nonnegative real valued, depending on the application. Reconstruction of the object $f(x)$ from $|F(u)|$ is equivalent to reconstruction of the Fourier phase $\psi(u)$ from $|F(u)|$ (hence the name phase retrieval), and reconstruction from $|F(u)|$ is equivalent to reconstruction from the object's autocorrelation

$$r(x) = \int f(x')f^*(x'-x)\mathrm{d}x' = \mathcal{F}^{-1}[|F|^2], \qquad (2)$$

since the autocorrelation is directly computable from $|F(u)|$.

At present the phase-retrieval algorithm that best combines the advantages of generality, noise tolerance, and computational efficiency is the iterative transform algorithm.[1-3] That (or any other) algorithm requires sufficiently strong object-domain constraints to ensure solution uniqueness and to achieve convergence to a solution within a reasonable number of iterations. The two constraints most often found to be both physically pertinent and useful to the iterative transform algorithm are nonnegativity and object-support constraints. The support of an object is the smallest closed set outside which the object is zero. Often one does not know *a priori* the support of an object but may know an upper bound on the support. In that case one would use the upper bound as a support constraint during the iterations. Tighter (smaller) support constraints typically result in faster convergence of the algorithm. This is particularly true for the case of complex-valued objects for which nonnegativity does not apply and the support constraint may be the only constraint.[4] In fact, without a tight support constraint the reconstruction of complex-valued images is extremely difficult.[4,5] Even with a perfectly tight support constraint such reconstructions are extremely difficult unless the object's support is within one of a restricted class of advantageous supports.[4]

Methods for estimating the support of the object from the support of the autocorrelation are given in Ref. 6 and in a previous paper.[7] Specifically, Ref. 6 describes a method applicable only to discrete objects (defined on a grid), and Ref. 7 describes methods applicable to both continuous convex objects and discrete objects with nonredundant spacings. Also, in Ref. 7 methods for determining upper bounds on the support of an object from the support of its autocorrelation are described. These upper bounds, called locator sets, contain a translation of every support consistent with the autocorrelation support. Since the twin image, $f^*(-x)$, has the same autocorrelation as $f(x)$, the locator sets have to be large enough to contain a translation of the support of the twin image as well as a translation of the support of the object.

In this paper we present methods for generating tighter upper bounds on the object's support, which we call single-sided locator sets. They are required to contain a translation of the support of any object or its twin image (but not necessarily both) that would give rise to a given autocorrelation support. These methods are based on the geometry of the autocorrelation support. Only one rule, stated in Theorem 6 below, is applicable to all situations. However there are a large variety of geometries that are covered by more than one method, so that for a particular object there is usually a fairly tight single-sided locator set.

Section 2 of this paper establishes definitions and notation and comments on the support of an autocorrelation for the case of complex-valued objects. Section 3 contains the methods for constructing the bounds on the object's support, stated in terms of theorems and corollaries with examples. The proofs of the theorems and corollaries are in Appendix A. Section 4 shows an image-reconstruction example that makes use of the support upper bounds as support constraints. Section 5 contains a summary and conclusions.

2. DEFINITIONS, NOTATION, AND BACKGROUND

Before setting out the rules for determining single-sided locator sets, we need some definitions and notation. First,

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in order to carry out a thorough analysis, we must specify the class of object representations. A natural class of representations to consider, though not the most general,[8] is all linear combinations of delta functions and compactly supported complex square-integrable functions. Thus an object representation, f, can be written as

$$f(x) = \sum_{1}^{N} \alpha_i \delta(x - x_i) + f_c(x), \tag{3}$$

where $\{x_i\}_1^N \subset \mathbf{R}^2$, $\delta(x - x_i)$ is the delta function at the point x_i, $\{\alpha_i\}_1^N \subset \mathbf{C}\backslash\{0\}$, and f_c is a complex square-integrable function with compact support. (A set is compact if it is closed and bounded, i.e., of finite extent.) Here $\subset$ denotes a subset (not necessarily strict). For an object representation f as above, the support, S_f, is defined by

$$S_f = \cup_1^N\{x_i\} \cup \operatorname{supp}(f_c), \tag{4}$$

where $\operatorname{supp}(f_c)$ is the support of f_c and is defined to be the smallest closed set outside which f_c is zero almost everywhere. The autocorrelation of f, $f \star f$, is just the convolution of f with its conjugate reflection. In general the support $S_{f\star f}$ of $f \star f$ satisfies that

$$S_{f\star f} \subset S_f - S_f = \{x - y{:}x, y \in S_f\}. \tag{5}$$

In two special cases, we actually have equality in Eq. (5). The first case is when f is real nonnegative.[7] The second case is when S_f and $S_{f\star f}$ are both convex. The latter result is a special case of the well-known Titchmarsh–Lions theorem in distribution theory.[9] It can also be shown that if one considers the subclass of object representations supported by the integer grid, $\mathbf{Z}^2$, then $S_f - S_f = S_{f\star f}$ with probability 1 (although counterexamples can be constructed). The viewpoint of this paper will be to assume that $S_{f\star f} = S_f - S_f$, even though we are in the continuous-support case. Henceforth we drop the f and look at compact sets in $\mathbf{R}^2$, which will be the supports of two-dimensional objects. The goal is to find methods or rules that give single-sided locator sets that are as tight as possible for the object support S based only on observing $S - S$.

Definitions. Let $\mathcal{S}$ be the class of all nonempty compact sets in $\mathbf{R}^2$. A set A is an autocorrelation support if and only if there exist $S \in \mathcal{S}$ such that $S - S = A$. For any autocorrelation support A, we say that a set $S \in \mathcal{S}$ generates A if $A = S - S$, and we denote the class of all such generating sets by $\mathcal{S}(A)$.

We say that compact supports S_0 and S_1 are equivalent, and we write $S_0 \sim S_1$, if there exists an $x \in \mathbf{R}^2$ such that $S_0 + x = S_1$ or $-S_0 + x = S_1$. Note that $S_0 \sim S_1$ implies that they generate the same autocorrelation support A, i.e., $S_0 - S_0 = S_1 - S_1$. We are using the definitions $-S = \{-x{:}x \in S\}$ and $S + x = \{y + x{:}y \in S\}$.[7]

Let S_0 and S_1 be two compact supports. Then we say that S_0 is dominated by S_1 if there exists an $S_1' \subset S_1$ such that $S_0 \sim S_1'$. If $\mathcal{S}_0$ and $\mathcal{S}_1$ are both classes of compact supports, we say that $\mathcal{S}_0$ is dominated by $\mathcal{S}_1$ if for all $S_0 \in \mathcal{S}_0$ there exist $S_1 \in \mathcal{S}_1$ such that S_0 is dominated by S_1.

Let A be an autocorrelation support. We say that a set $L \subset \mathbf{R}^2$ is a locator set for A if L is compact and it contains a translation of every compact support that generates A. We say that a set $L \subset \mathbf{R}^2$ is a single-sided locator set for A if L is compact and it contains a translation of S or $-S$ (but not necessarily of both) for all supports S that generate A.

3. METHODS OF DETERMINING SINGLE-SIDED LOCATOR SETS

In Ref. 7 it was shown that, if A is an autocorrelation support and $a \in A$, then $A \cap (A + a)$ is a locator set for A. It is natural, for the sake of tightness, to choose an element a from the boundary of A. Our approach to finding single-sided locator sets is to start with $A \cap (A + a)$ and intersect it with more translations of A or else with a properly located half-plane. Most of the rules for finding single-sided locator sets are based on the former intersection and are fundamentally based on the following theorem and its corollary. The theorem originally appeared as a result in Eq. (13) of Ref. 7, which was given there as rule for finding locator sets. However, this statement was false, and the rule actually finds single-sided locator sets, not locator sets. It should be noted that this misstatement in Ref. 7 does not affect any of the other results in that paper.

Theorem 1. Let A be an autocorrelation support, let $\mathcal{S}_0$ be a class of supports that generate A, and suppose that $\mathcal{S}(A)$ is dominated by $\mathcal{S}_0$. Let B be a nonempty set such that

$$B \subset \cap\{S_0{:}S_0 \in \mathcal{S}_0\}. \tag{6}$$

Then $L = \cap\{A + b{:}b \in B\}$ is a single-sided locator set for A.

Corollary 1. Let A be an autocorrelation support and B be a nonempty set such that, for all supports S that generate A,

$$B \subset S - x \quad \text{or} \quad B \subset x - S \qquad \text{for some } x \in \mathbf{R}^2. \tag{7}$$

Then $L = \{A + b{:}b \in B\}$ is a single-sided locator set for A.

The essential idea of the proof of Theorem 1 was given in Ref. 7, and for completeness it is reproduced in Appendix A. In order to generate single-sided locator sets, we want to find rules, which use only the knowledge of the autocorrelation support A, for determining sets B that satisfy the hypothesis of Corollary 1. The basis of this determination will be to investigate the geometry of the maximal points (defined below) in the autocorrelation support A. Different geometries will give rise to different sets B and hence to different single-sided locator sets. First we need to define precisely what we mean by maximal points and locally maximal points.

Definitions. A vector $u \in \mathbf{R}^2$ is a unit vector if $|u| = 1$. The inner product of two vectors $x, y \in \mathbf{R}^2$ is denoted by $\langle x, y\rangle$. Now let B be an arbitrary compact set in $\mathbf{R}^2$. The set of maximal points in the u direction, $E(B, u)$, are the points $x \in B$ such that $\langle x, u\rangle \geq \langle y, u\rangle$ for all $y \in B$. A neighborhood of a point $x \in \mathbf{R}^2$ is any disk of positive radius centered at x. The set of all locally maximal points in the u direction, $E_l(B, u)$, are the points $x \in B$ for which there exists a neighborhood V_x of x such that $\langle x, u\rangle \geq \langle y, u\rangle$ for all $y \in V_x \cap B$.

Note that $E(B, u)$ represents, in some sense, the points in B that are the farthest out in the u direction, and $E_l(B, u)$ represents the points in B that are locally the farthest out in the u direction. Also note that $E(B, u) \subset E_l(B, u)$. Figure 1 illustrates these definitions. Let A be an autocorrelation support, and let S be a support that generates A, i.e., $S - S = A$. Then there are some important relationships among

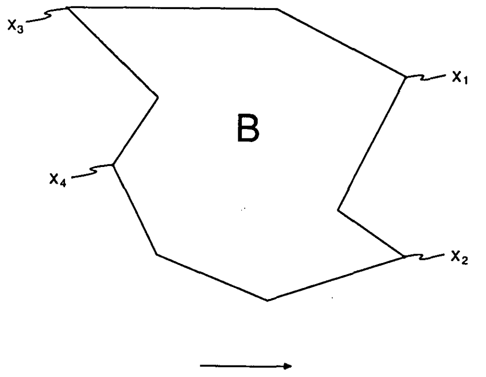

Fig. 1. Example of maximal and locally maximal points. $E(B, u) = \{x_1, x_2\}$, $E_l(B, u) = \{x_1, x_2\}$, $E(B, -u) = \{x_3\}$, and $E_l(b, -u) = \{x_3, x_4\}$.

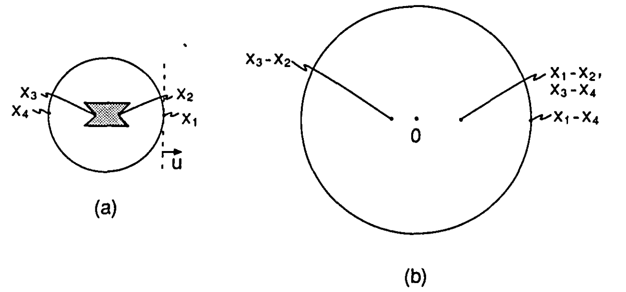

Fig. 2. Example of a set S such that $E_l(S, u) - E_l(S, -u) \neq E_l(A, u)$. (a) Set S, where $E(S, u) = \{x_1\}$, $E(S, -u) = \{x_4\}$, $E_l(S, u) = \{x_1, x_3\}$, and $E_l(S, -u) = \{x_2, x_4\}$ (the shaded area is not considered to be part of S); (b) set $A = S - S$, which is a disk, where $E(A, u) = \{x_1 - x_4\} = E_l(A, u)$ and $E_l(S, u) - E_l(S, -u) = \{x_1 - x_2, x_1 - x_4, x_3 - x_2, x_3 - x_4\}$, which is not equal to $E_l(A, u)$.

$E(A, u)$, $E(S, u)$, and $E(S, -u)$. In a similar manner, there are some important relationships among $E_l(A, u)$, $E_l(S, u)$, and $E_l(S, -u)$. These are outlined in the next result, Theorem 2.

Theorem 2. Let u be a unit vector, A be an autocorrelation support, and S be a support that generates A. Then

$$E(A, u) = E(S, u) - E(S, -u) \tag{8}$$

and

$$E_l(A, u) \subset E_l(S, u) - E_l(S, -u). \tag{9}$$

The inclusion in formula (9) cannot be strengthened to equality, as demonstrated by the example in Fig. 2.

Based on Corollary 1 and Theorem 2, we investigate mainly geometric conditions on $E(A, u)$ that imply that there exist $B \subset E(A, u)$ so that for all supports S generating A we have

$$B \subset x - E(S, -u) \qquad \text{for some } x \in E(S, u) \tag{10}$$

or

$$B \subset E(S, u) - x \qquad \text{for some } x \in E(S, -u). \tag{11}$$

Corollary 1 says that such a set B defines a single-sided locator set for A. We also investigate geometric conditions on $E_l(A, u)$ that would imply the existence of a set B satisfying analogous conditions. Basically, the maximal points in S, which stick out most in a given direction, cause corresponding sets of points in A that stick out. Our approach is to use the points in A that stick out to infer existence of the maximal points in S, which we can then use to form B to use in Corollary 1 to define a single-sided locator set.

Definitions. Let v be a unit vector and $c \in \mathbf{R}$. Then the half-plane above cv, $H^+(v, c)$, is defined by $H^+(v, c) = \{x \in \mathbf{R}^2: \langle x, v\rangle \geq c\}$. If u and v are both unit vectors, then we say that u is perpendicular to v, and write $u \perp v$, if $\langle u, v\rangle = 0$.

Based on Theorems 1 and 2, we are now in a position to state precisely a set of rules for finding single-sided locator sets. As stated above, different rules are applicable depending on the various geometries of the set $E(A, u)$ and/or $E_l(A, u)$, with the idea that we would like to handle as many general cases as possible. To outline the cases, we will state rules that are applicable to the following geometries:

(i) $E(A, u) \cap H^+(v, c)$ consists of two points, where $c \in \mathbf{R}$ and u and v are unit vectors such that $u \perp v$ [i.e., consists of an endpoint of $E(A, u)$ and its nearest neighbor in $E(A, u)$].

(ii) $E(A, u)$ consists of two points.

(iii) $E(A, u)$ is totally asymmetric.

(iv) $E(A, u)$ consists of three points.

(v) $E(A, u)$ is a line segment, and A satisfies a convexity condition.

(vi) $E(A, u)$ is a single point.

(vii) $E_l(A, u)$ has a special form.

We will now state each rule precisely in the form of theorems and corollaries and show examples.

Theorem 3. Let A be an autocorrelation support, u and v be perpendicular unit vectors, and $c \in \mathbf{R}$. Suppose that $E(A, u) \cap H^+(v, c) = \{a_1, a_2\}$, where a_1 and a_2 are assumed to be distinct. Then $L = A \cap (A + a_1) \cap (A + a_2)$ is a single-sided locator set for A.

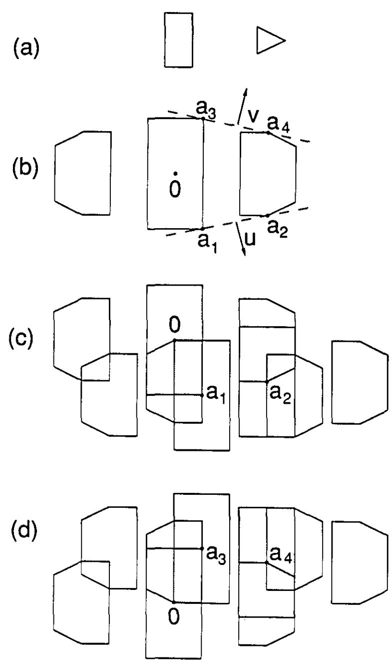

Fig. 3. Example of the two-point rule. (a) Set S, (b) set $A = S - S$ with $E(A, u) = \{a_1, a_2\}$ and $E(A, v) = \{a_3, a_4\}$, (c) single-sided locator set $L_1 = A \cap (A + a_1) \cap (A + a_2)$, (d) single-sided locator set $L_2 = A \cap (A + a_3) \cap (A + a_4)$ (indicated by the shaded areas).

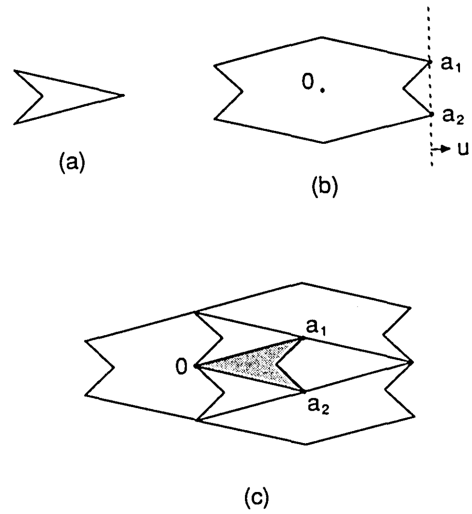

Fig. 4. Example of the two-point rule. (a) Set S, (b) set $A = S - S$ and $E(A, u) = \{a_1, a_2\}$, (c) shaded area is the single-sided locator set $L = A \cap (A + a_1) \cap (A + a_2)$, which is identical to a translation of $-S$.

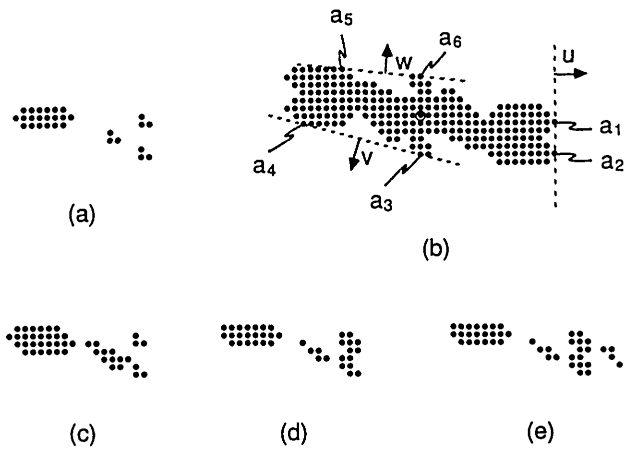

Fig. 5. Example of the two-point rule. (a) Set S consisting of discrete points; (b) set $A = S - S$ with $E(A, u) = \{a_1, a_2\}$, $E(A, v) = \{a_3, a_4\}$, and $E(A, w) = \{a_5, a_6\}$; (c) single-sided locator set $L_1 = A \cap (A + a_1) \cap (A + a_2)$; (d) single-sided locator set $L_2 = A \cap (A + a_3) \cap (A + a_4)$; (e) single-sided locator set $L_3 = A \cap (A + a_5) \cap (A + a_6)$.

An immediate corollary of Theorem 3 is the two-point rule, i.e., if $E(A, u)$ consists of only two points a_1 and a_2, then L, as given in Theorem 3, is a single-sided locator set. We now state this result precisely.

Corollary 2 (*Two-Point Rule*). Let A be an autocorrelation support and u be a unit vector. If $E(A, u) = \{a_1, a_2\}$, then $L = A \cap (A + a_1) \cap (A + a_2)$ is a single-sided locator set for A.

We give illustrative examples of the two-point rule in Figs. 3–5. In Fig. 5 the dots represent single points located at the centers of the circles. Both Figs. 3 and 5 show the possibility of having multiple single-sided locator sets. These examples also tempt one to see whether there might be a way to combine the various single-sided locator sets. This is possible, and it is a special case of a more general result given below (Theorem 8). Figure 4 shows the possibility of a single-sided locator set's being equivalent to S, thus demonstrating that unique reconstructions of the support are sometimes possible.

We now want to state a rule, as Theorem 4, that is applicable when the set $E(A, u)$ satisfies a kind of total asymmetry. Combining this result with Theorem 3 will give a general result applicable to the case when the set $E(A, u)$ consists of three points. This is precisely stated in Corollary 3. However, before stating the two results, we need to define the concept of endpoints of the set $E(A, u)$.

Definitions. Let u and v be perpendicular unit vectors and B be a compact set. We say that b_1 and b_2 are the endpoints of $E(B, u)$ if $\langle b_1, v\rangle \le \langle x, v\rangle \le \langle b_2, v\rangle$ for all $x \in E(B, u)$. Also, if b_2 is as above, we say that b_2 is the v-positive endpoint of $E(B, u)$.

Suppose that $x, y \in \mathbf{R}^2$. Then the open and closed intervals, (x, y) and $[x, y]$, respectively, are defined by

$$(x, y) = \begin{Bmatrix} \{z{:}z = tx + (1 - t)y, t \in (0, 1)\} & \text{if } x \neq y \\ \varnothing & \text{if } x = y \end{Bmatrix} \tag{12}$$

and

$$[x, y] = \{z{:}z = tx + (1 - t)y, t \in [0, 1]\}, \tag{13}$$

where $\varnothing$ denotes the empty set.

The midpoint of x and y is defined as the point $(x + y)/2$.

Finally, if B_1 and B_2 are two subsets of $\mathbf{R}^2$, then $B_1 \backslash B_2$ denotes those points in B_1 that are not in B_2.

Theorem 4 (*Totally Asymmetric Rule*). Let u be a unit vector and A be an autocorrelation support. Suppose that a_1 and a_2 are two distinct endpoints of $E(A, u)$. If

$$a_1 + a_2 \notin [E(A, u)\backslash\{a_1, a_2\}] + [E(A, u)\backslash\{a_1, a_2\}], \tag{14}$$

then $L = A \cap (\cap\{A + a{:}a \in E(A, u)\})$ is a single-sided locator set for A.

Note that condition (14) is equivalent to stating that the midpoint m of a_1 and a_2 is not contained in $E(A, u)$ and that the only two points in $E(A, u)$ having m as a midpoint are a_1 and a_2; hence the heuristic terminology of total asymmetry. We now give a corollary that follows from Theorems 3 and 4 and covers the case when $E(A, u)$ consists of three points.

Corollary 3 (*Three-Point Rule*). Let A be an autocorrelation support and u be a unit vector. Suppose that $E(A, u) = \{a_1, a_2, a_3\}$, where $a_2 \in (a_1, a_3)$. Then $L_1 = A \cap (A + a_1) \cap (A + a_2)$ and $L_2 = A \cap (A + a_2) \cap (A + a_3)$ are single-sided locator sets for A. If in addition a_2 is not the midpoint of a_1 and a_3, then $L_3 = A \cap (A + a_1) \cap (A + a_2) \cap (A + a_3)$ is a single-sided locator set for A.

In Figs. 6 and 7 we give illustrative examples of the three-point rule and the totally asymmetric rule.

Now suppose that $E(A, u)$ is a continuous line segment. None of the previous rules applies, so in this case we are in need of a new rule. Shortly we will state a result that says that a single-sided locator set may be derived by intersecting A with translations of A to an endpoint and a midpoint of $E(A, u)$ respectively, provided that A satisfies a convexity condition. We first define the convexity condition and then give an illustrative example. We then state the rule precisely in Theorem 5.

Definition. Let B be a compact set and u be a unit vector. Then B is u convex if $x, y \in B$, and $x - y \perp u$ implies that $[x, y] \subset B$.

We give an illustrative example of u convexity in Fig. 8.

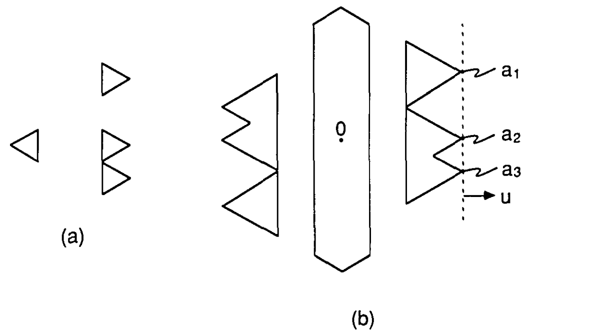

Fig. 6. Example of the asymmetric three-point rule. (a) Set S, (b) set $A = S - S$ and $E(A, u) = \{a_1, a_2, a_3\}$. In this case the single-sided locator set $L = A \cap (A + a_1) \cap (A + a_2) \cap (A + a_3)$, based on the three-point rule where the three points are asymmetric, is identical to a translation of the original support S.

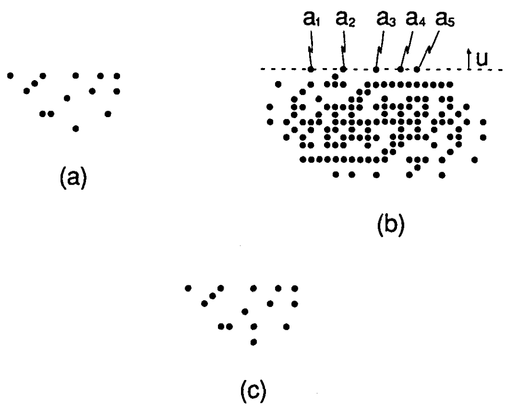

Fig. 7. Example of the totally asymmetric rule. (a) Set S consisting of discrete points, (b) set $A = S - S$ and $E(A, u) = \{a_1, a_2, a_3, a_4, a_5\}$, (c) a single-sided locator set $L = A \cap [\cap_1^5 (A + a_i)]$ determined by the asymmetric rule. Note that L is the same as a translation of the original support S except for one extra point.

Theorem 5 (u-Convex Rule). Let A be an autocorrelation support and u be a unit vector. Let a_1 and a_2 be the endpoints of $E(A, u)$, and let m be the midpoint of a_1 and a_2. If A is u convex, then $L_1 = A \cap (A + a_1) \cap (A + m)$ and $L_2 = A \cap (A + m) \cap (A + a_2)$ are single-sided locator sets for A.

An example of the u-convex rule is given in Fig. 9. A natural question is whether the u-convexity condition is really needed in the hypothesis in Theorem 5; i.e., is the result true under the weaker assumption that $E(A, u) = [a_1, a_2]$? This question is answered in a counterexample illustrated in Fig. 10. In this example, $E(A, u)$ is a line segment, A is not u convex, and the set L_1, as defined in Theorem 5, is not a single-sided locator set. Thus the two sets L_1 and L_2, as given in Theorem 5, are not necessarily single-sided locator sets based only on observing that $E(A, u)$ is a line segment.

We now give a rule that is based on $E(A, u)$ consisting of a single point. In this case we can determine a single-sided locator set for A by intersecting A with $A + E(A, u)$ and an appropriately located half-plane. Before stating the rule in Theorem 6, we need a definition and some notation.

Definition. Let u be a unit vector and let B be a compact set. Then the diameter of B in the u direction, $d(B, u)$, is defined by

$$d(B, u) = \sup\{\langle x - y, u\rangle : x, y \in B\}, \tag{15}$$

where sup denotes supremum.

Theorem 6. Let u and v be unit vectors such that $u \neq \pm v$, and let A be an autocorrelation support. Suppose that $E(A, u) = \{a\}$. Let $d = d(A, v)$ and $H = H^+(v, -d/4) + a/2$. Then $L = A \cap (A + a) \cap H$ is a single-sided locator set.

An example of this rule is given in Fig. 11. In this example, the object support, S, is a disk, and the autocorrelation support generated by S is a disk with a diameter twice that of S. It can be shown that, for the disk, the single-sided locator set determined by the rule in Theorem 6 and displayed in Fig. 11 is a minimal closed single-sided locator set, i.e., there does not exist any closed proper subset of this single-sided locator set that is itself a single-sided locator set for A.

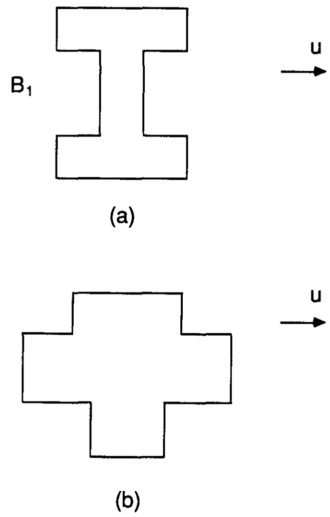

Fig. 8. Examples of u-convex and non-u-convex sets. (a) Set B_1 is not u convex, (b) set B_2 is u convex.

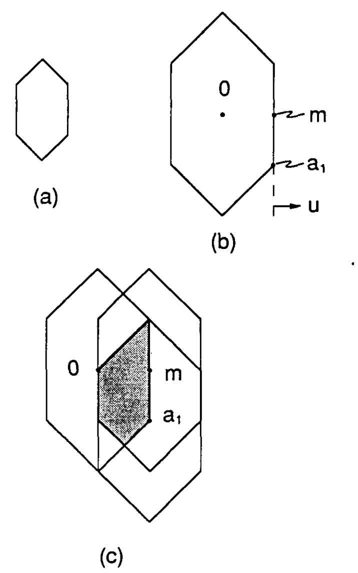

Fig. 9. Example of the u-convex rule. (a) Set S; (b) set $A = S - S$, where a_1 and m are an endpoint and the midpoint, respectively, of $E(A, u)$; (c) a single-sided locator set $L = A \cap (A + a_1) \cap (A + m)$ determined by the u-convex rule (indicated by the shaded area).

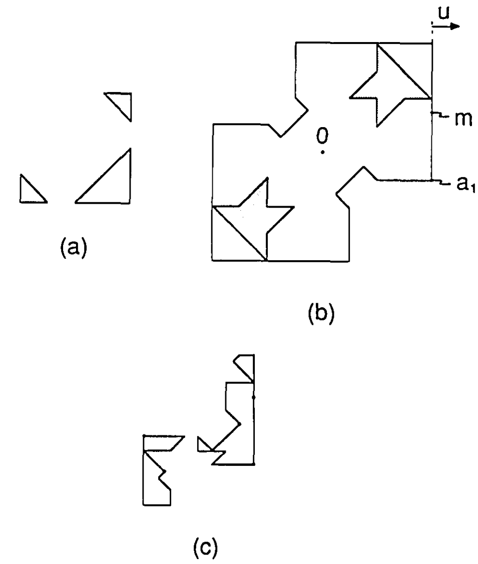

Fig. 10. Example showing the insufficiency of $E(A, u) = [a_1, a_2]$. (a) Set S; (b) set $A = S - S$, where a_1 and m are an endpoint and the midpoint, respectively, of $E(A, u) = [a_1, a_2]$ (the shaded regions, which are open sets, are not part of A); (c) set $L = A \cap (A + m) \cap (A + a_1)$. Note that L is not a single-sided locator set since no translation of S or $-S$ is contained in L.

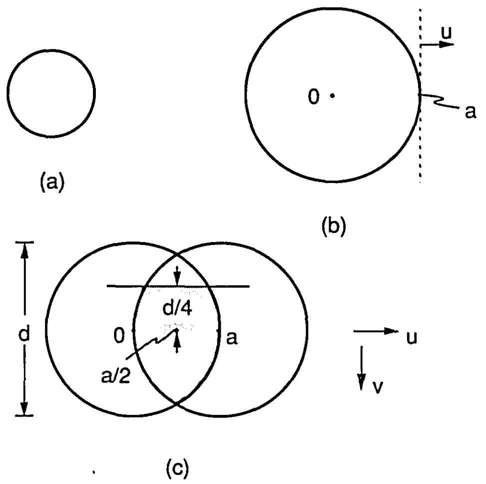

Fig. 11. Example of the one-point rule. (a) Set S; (b) set $A = S - S$ and $E(A, u) = \{a\}$; (c) single-sided locator set $L = A \cap (A + a) \cap H$, where H is determined by the displayed u, v, and the one-point rule as in Theorem 6.

Up to this point we have only stated rules based on the maximal points $E(A, u)$. We now want to state a rule based on the geometry of the locally maximal points, $E_l(A, u)$. Before stating the result in Theorem 7, we need a definition and some notation.

Definition. Let B be a compact set, and let $b_1, b_2, b_3, b_4 \in B$. Then we denote the ordered 4-tuplet of these points by (b_1, b_2, b_3, b_4). Note that the ordering is important, e.g., $(b_1, b_2, b_3, b_4) = (b_2, b_1, b_3, b_4)$ if and only if $b_1 = b_2$. The set of all parallelograms in B, $\mathcal{P}(B)$, is defined as

$$\mathcal{P}(B) = \{(b_1, b_2, b_3, b_4): b_i \in B \text{ for } i = 1, 2, 3, 4; \quad b_1 \neq b_j \text{ for } j = 2, 3, 4; \text{ and } b_1 + b_3 = b_2 + b_4\}. \quad (16)$$

Note that $\mathcal{P}(B)$ is really all ordered 4-tuplets of points that are in the set B and are vertices of a parallelogram. We do allow the case of three distinct collinear points to be in $\mathcal{P}(B)$, say, $\{x_1, x_2, x_3\}$, where x_2 is the midpoint of x_1 and x_3. In this case the representation would have to be, for example, (x_1, x_2, x_3, x_2) and not (x_2, x_1, x_2, x_3). Note that in the case of four distinct collinear points x_1, x_2, x_3, x_4, where $x_1 + x_3 = x_2 + x_4$, then (x_1, x_2, x_3, x_4) and (x_2, x_1, x_4, x_3) are both in $\mathcal{P}(B)$.

Theorem 7. Let u be a unit vector and A be an autocorrelation support. Let $a' \in E_l(A, u) \backslash E(A, u)$ and $c = \langle a', u \rangle$. Let $I = A \cap H^+(u, c)$; if

$$\mathcal{P}' = \{(a_1, a_2, a_3, a_4) \in \mathcal{P}(I): a_1 \in E(A, u), a_3 = a'\} = \varnothing, \quad (17)$$

then $L = A \cap (\cap\{A + a: a \in E(A, u)\}) \cap (A + a')$ is a single-sided locator set.

An example of the rule stated in Theorem 7 is given in Fig. 12. Note that, by using Theorem 7, a support equivalent to S was reconstructed exactly, whereas when Theorem 4 was used in Fig. 7 there was one additional point.

We now state a result about combining two or more single-sided locator sets in order to derive a new single-sided locator set that is tighter than both of the previous single-sided locator sets. Intuitively, if we had two single-sided locator sets L_1 and L_2, we would like to intersect L_1 with a translation of L_2 or $-L_2$. The reason why we need to consider $-L_2$ is that only a translation of S or $-S$ must be contained in L_1 and L_2. However, it may be S for L_1 and $-S$ for L_2 or vice versa. Thus there is a need for a rule for determining, from the original set A and the two single-sided locator sets, whether to intersect L_1 with a translation of L_2 or $-L_2$ and what the translation ought to be. We now state such a rule in Theorem 8. However, before this, we again need a definition.

Definition. Let u and v be unit vectors such that $u \neq \pm v$, and let B be a compact set. We say that B is centered relative to u and v if

$$\sup\{\langle x, u\rangle : x \in B\} = -\inf\{\langle x, u\rangle : x \in B\} = \frac{d(B, u)}{2} \quad (18)$$

and

$$\sup\{\langle x, v\rangle : x \in B\} = -\inf\{\langle x, v\rangle : x \in B\} = \frac{d(B, v)}{2}, \quad (19)$$

where inf denotes infimum.

Note that, if B is centered relative to u and v, then any nonzero translation of B is not centered. Hence for any

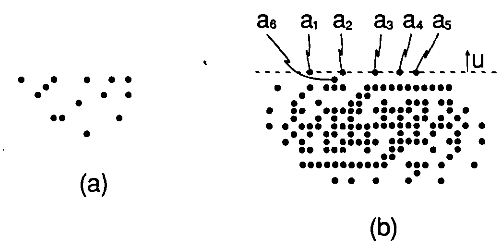

Fig. 12. Example of the rule in Theorem 7. (a) Set S consisting of discrete points [same as Fig. 7(a)]; (b) set $A = S - S$ and $E(A, u) = \{a_1, a_2, a_3, a_4, a_5\}$, where $a' = a_6 \in E_l(A, u)$ [actually all points in A are in $E_l(A, u)$]. The single-sided locator set $L = A \cap [\cap_1^5(A + a_i)]$, as determined by the rule in Theorem 7, is identical to a translation of the original object support shown in (a).

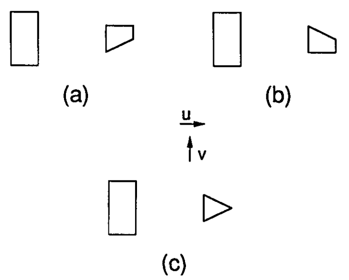

Fig. 13. Example of combining single-side locator sets. (a), (b) Single-sided locator sets from Fig. 3; (c) a single-sided locator set L obtained by combining the single-sided locator sets in (a) and (b), using Theorem 8. Note that the single-sided locator set is identical to a translation of the original support S, shown in Fig. 3(a).

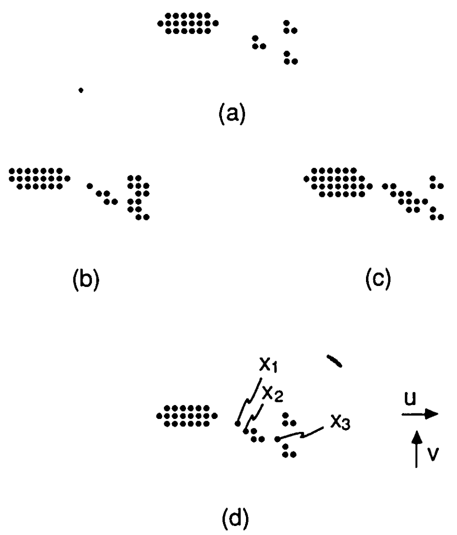

Fig. 14. Example of combining single-sided locator sets. (a) Set S, consisting of discrete points (same as in Fig. 5); (b), (c) single-sided locator sets L_2 and L_1 from Fig. 5; (d) a single-sided locator set L obtained by combining L_1 and L_2, using Theorem 8. Note that L contains only three more points, x_1, x_2, and x_3, than the original support S. Since $L\backslash\{x_1\} - L\backslash\{x_1\} = A$, i.e., $L\backslash\{x_1\}$ generates A, the points x_2 and x_3 must be contained in any single-sided locator set.

compact set there is one and only one translation that makes it centered.

Theorem 8. Let A be an autocorrelation support, L_1 and L_2 be single-sided locator sets for A, and u and v be unit vectors such that $u \neq \pm v$. Assume that L_1 and L_2 are centered relative to u and v, $d(L_1, u) = d(L_2, u) = d(A, u)/2$, and $d(L_1, v) = d(L_2, v) = d(A, v)/2$. Let $L_{12} = L_1 \cap L_2$ and $L_{12}^- = L_1 \cap (-L_2)$. Then

(a) Either $A \subset L_{12} - L_{12}$ or $A \subset L_{12}^- - L_{12}^-$;

(b) If $A \not\subset L_{12} - L_{12}$, then L_{12}^- is a single-sided locator set for A; and

(c) If $A \not\subset L_{12}^- - L_{12}^-$, then L_{12} is a single-sided locator set for A.

In order to use the above rule as it now stands, one must check by trial and error for various combinations of u and v to see whether the hypothesis of Theorem 8 is satisfied. Examples are given in Figs. 13 and 14. As the examples show, this theorem may be useful in some situations. Note that, if L is a single-sided locator set generated from L_1 and L_2 by Theorem 8, then Theorem 8 might possibly be applied again, using a new unit vector w, with $\pm u \neq w \neq \pm v$ to yield a tighter single-sided locator set.

4. IMAGE-RECONSTRUCTION EXAMPLE

Figure 15 shows an example of employing the two-point rule (Corollary 2) to produce a support constraint for use with the iterative Fourier-transform algorithm to reconstruct an object from the modulus of its Fourier transform. Figure 15(A) shows the object, which consists of two separated parts: an image of a satellite model and a narrow triangle below it. In this case the image is known to be real valued and nonnegative. The modulus of its Fourier transform, shown in Fig. 15(B), is the only datum assumed to be known. Figure 15(C) shows an estimate of the support of the object's autocorrelation, obtained by inverse Fourier transforming the squared Fourier modulus and thresholding the result at 0.005 times its peak value. Arrows indicate the two points on the edge of the estimated autocorrelation support used in the triple intersection to obtain the single-sided locator set, which is shown in Fig. 15(D). Note that this single-sided locator set is suggestive of the actual support of the object. The iterative transform algorithm was employed to reconstruct the object from its Fourier modulus, using a nonnegativity constraint and the single-sided locator set estimate shown in Fig. 15(D) as a support constraint. The initial estimate of the object was the single-sided locator set filled with uniformly distributed nonnegative real numbers. Ten iterations of the hybrid input–output version of the algorithm (with a feedback parameter of $\beta = 0.7$) resulted in the very good image shown in Fig. 15(E). Further progress was impeded by the fact that the estimated support constraint was smaller than the true support of the object owing to the finite-threshold value used to estimate the autocorrelation support. Next the initial support constraint was enlarged twice, each time by adding to the support any pixel whose nearest neighbor was within the support. Enlarging of the support constraint is performed in order to ensure that the constraint is not inadvertently truncating part of the object.[3] Another ten iterations were performed, using the enlarged support constraint, which is shown in Fig. 15(F). The reconstructed image, shown in Fig. 15(G), is a more faithful representation of the object.

The small number of iterations required to reconstruct the object demonstrates the power of the single-sided locator sets as support constraints for objects with separated parts. By comparison, several dozen iterations are typically required for real nonnegative objects.[2]

When diffraction effects are also included in the data, then sidelobes of the impulse response require a larger threshold to be used in the estimation of the autocorrelation support in order to avoid counting sidelobes that are outside the autocorrelation support as being within the autocorrelation support. Then it is advantageous to employ a weighting (apodization) function in the Fourier domain to reduce sidelobes, even though it results in a loss of resolution in the autocorrelation.

When one is dealing with complex-valued objects,[4] the autocorrelation will also be complex valued and hence speckled. A thresholding operation on the autocorrelation

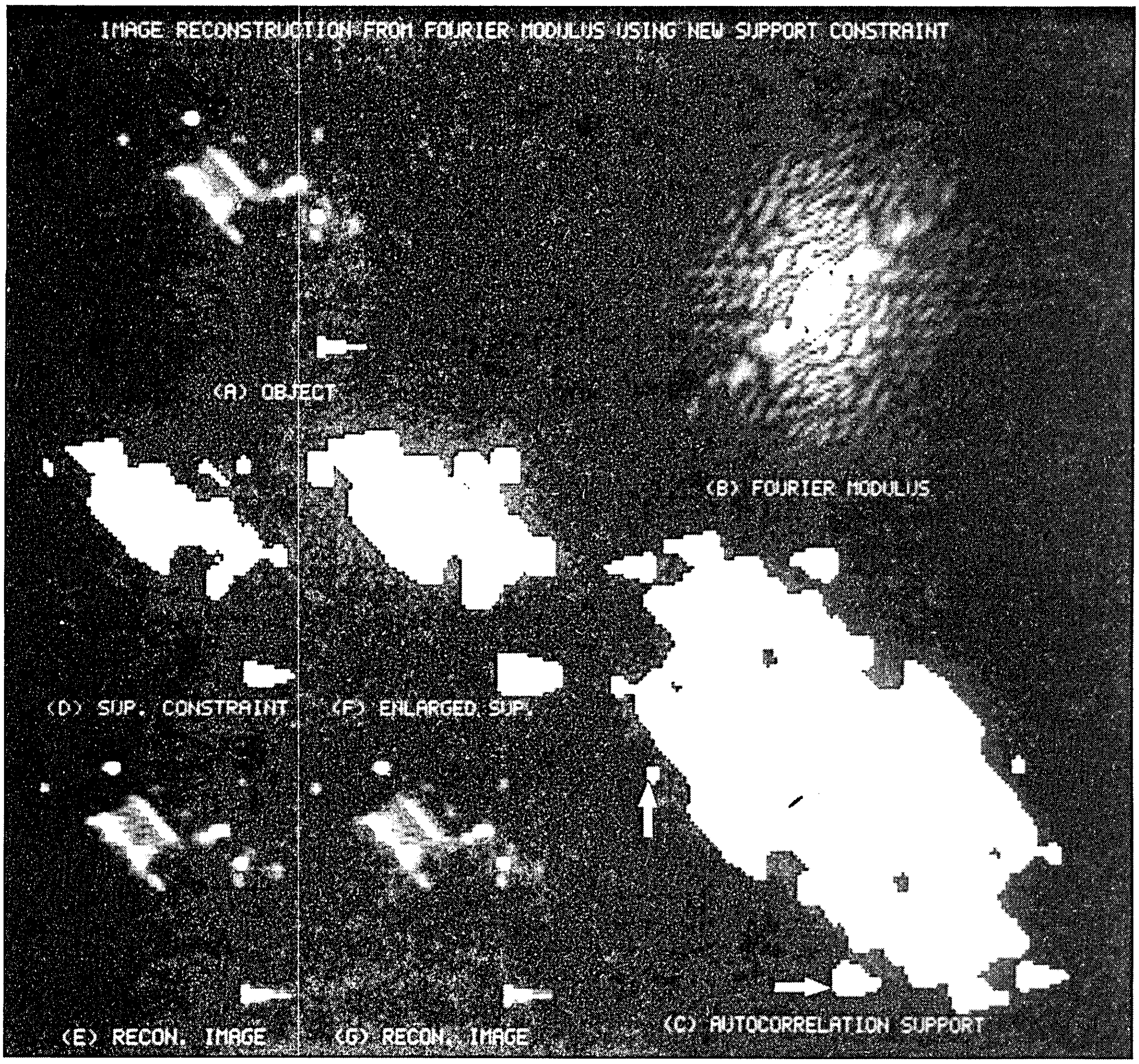

Fig. 15. Computer reconstruction example using single-sided locator set. (A) Object, (B) Fourier modulus, (C) thresholded autocorrelation (estimated autocorrelation support), (D) single-sided locator set computed from (C), (E) image reconstructed from Fourier modulus with ten iterations of the iterative Fourier-transform algorithm using the single-sided locator set in (D) as a support constraint, (F) enlarged support constraint, (G) image reconstructed from ten more iterations using (F) as a support constraint.

magnitude then causes locations where speckle nulls appear to be counted as outside the autocorrelation support. This can easily be remedied by, for example, (1) convolving the threshold autocorrelation with a small block of two or three pixels' diameter and (2) rethresholding at an appropriate level.

5. CONCLUSIONS

Determination of a tight object support constraint from the autocorrelation function is useful for solving the phase-retrieval problem, i.e., reconstructing an object from the modulus of its Fourier transform. In this paper we have described several new rules for computing single-sided locator sets, which are upper bounds on the object support determined from the autocorrelation support. These rules generally form much tighter upper bounds on the object support than do previously described rules[7] for determining locator sets. In order to demonstrate its effectiveness, one of the rules was used to compute a single-sided locator set for a digitally simulated object. It was shown to be useful as a support constraint for reconstructing the object from the modulus of its Fourier transform by the iterative Fourier-transform algorithm. It speeds convergence of the algorithm and helps to avoid the twin-image stagnation problem.[3]

APPENDIX A

Proof of Theorem 1. Let S be a support generating A. There exist a support $S_0 \in \mathcal{S}_0$ and $x \in \mathbf{R}^2$ such that $S + x \subset S_0$ or $-S + x \subset S_0$. Suppose that the former is true. Let $y \in S_0$. Then

$$\begin{aligned} S + x &\subset S_0 \\ &= S_0 - y + y \\ &\subset S_0 - S_0 + y \\ &= A + y. \end{aligned} \tag{A1}$$

Thus, since formula (A1) is true for all $y \in S_0$ and since $B \subset S_0$,

$$S + x \subset \cap\{A + y{:}y \in S_0\} \subset \cap\{A + y{:}y \in B\} = L. \qquad \text{(A2)}$$

In the case $-S + x \subset S_0$, a similar argument shows that $-S + x \subset L$. The result now follows.

Proof of Corollary 1. Let S be a support generating A, and suppose that condition (7) is satisfied for some $x \in \mathbf{R}^2$. If $B \subset S - x$, let $S_0 = S - x$; and if $B \subset x - S$ but $B \not\subset S - x$, then let $S_0 = x - S$. Let $\mathcal{S}_0$ be the union of all possible S_0 generated in the above manner. Clearly B satisfies condition (6), and so, by Theorem 1, the result follows.

Proof of Theorem 2. Let $x, y \in S$, and suppose that $x \notin E(S, u)$ or $y \notin E(S, -u)$. Then there exists $s \in S$ such that $\langle s,u\rangle > \langle x,u\rangle$ or $\langle s,-u\rangle > \langle y,-u\rangle$. But this implies that $\langle s - y,u\rangle > \langle x - y,u\rangle$ or $\langle x - s,u\rangle > \langle x - y,u\rangle$ and hence that $x - y \notin E(A, u)$. Thus we have $E(A, u) \subset E(S, u) - E(S, -u)$.

Now let $x \in E(S, u)$ and $y \in E(S, -u)$. Then, if $x', y' \in S$, we have

$$\langle x - y, u\rangle = \langle x, u\rangle + \langle y, -u\rangle \geq \langle x', u\rangle + \langle y', -u\rangle = \langle x' - y', u\rangle. \qquad \text{(A3)}$$

Since $S - S = A$, this implies $x - y \in E(A, u)$. Thus $E(S, u) - E(S, -u) \subset E(A, u)$. By combining this result with the result in previous paragraph, we have the result stated in Eq. (8).

Let $x, y \in S$ and assume that $x \notin E_l(S, u)$. Let V be a neighborhood of $x - y$. Since $V + y$ is a neighborhood of x, there exists $z \in V$ such that $z + y \in S$ and $\langle z + y, u\rangle > \langle x, u\rangle$. But this implies that $\langle z, u\rangle > \langle x - y, u\rangle$. Since $z \in V \cap (S - y) \subset V \cap A$ and V is an arbitrary neighborhood of $x - y$, this implies that $x - y \notin E_l(A, u)$.

If $x, y \in S$ and $y \notin E_l(S, -u)$, a similar proof shows that $x - y \notin E_l(A, u)$. Thus $E_l(A, u) \subset E_l(S, u) - E_l(S, -u)$.

Proof of Theorem 3. Let S be a support that generates A. By Theorem 2, $a_1 = x_1 - y_1$ and $a_2 = x_2 - y_2$, where $x_1, x_2 \in E(S,u)$ and $y_1, y_2 \in E(S, -u)$. Since $a_1, a_2 \in E(A, u)$, $\langle a_1, u\rangle = \langle a_2, u\rangle$. This implies that $\langle a_1, v\rangle \neq \langle a_2, v\rangle$, since $a_j = \langle a_j, u\rangle u + \langle a_j, v\rangle v$ for $j = 1, 2$ and $a_1 \neq a_2$. Without loss of generality, we can assume that $\langle a_1, v\rangle > \langle a_2, v\rangle$.

Thus either $\langle x_1, v\rangle > \langle x_2, v\rangle$ or $\langle y_2, v\rangle > \langle y_1, v\rangle$. Suppose that the former is true. Then $\langle x_1 - y_2, v\rangle > \langle x_2 - y_2, v\rangle \geq c$, and $x_1 - y_2 \in E(A, u)$ by Theorem 2. Thus $x_1 - y_2 \in E(A, u) \cap H^+(v, c)$. Since $x_1 - y_2 \neq x_2 - y_2 = a_2$, we have $x_1 - y_2 = a_1$, which in turn implies that $y_2 = y_1$. Thus $\{0, a_1, a_2\} \subset S - y_1$. If $\langle y_2, v\rangle > \langle y_1, v\rangle$, a similar proof shows that $x_1 = x_2$ and in this case $\{0, a_1, a_2\} \subset x_1 - S$. By Corollary 1, the result follows.

Proof of Theorem 4. Let S be a support generating A, let x_1 and x_2 be the endpoints of $E(S, u)$, and let y_1 and y_2 be the endpoints of $E(S, -u)$. Let v be a unit vector perpendicular to u. Without loss of generality, we can assume that a_1, x_1, and y_2 are the v-positive endpoints. Then it is easy to see that $a_1 = x_1 - y_1$ and $a_2 = x_2 - y_2$. By Theorem 2, $x_1 - y_2 \in E(A, u)$ and $x_2 - y_1 \in E(A, u)$, and note that

$$(x_1 - y_2) + (x_2 - y_1) = a_1 + a_2. \qquad \text{(A4)}$$

Since a_1 and a_2 are the distinct endpoints of $E(A, u)$ and both summands on the left-hand side of Eq. (A4) are in $E(A, u)$ by Theorem 2, at least one of the two summands on the left-hand side of Eq. (23) is in $\{a_1, a_2\}$. Thus, by the hypothesis, $x_1 - y_2$ or $x_2 - y_1$ is equal to one of the endpoints a_1 or a_2.

If $x_1 - y_2 = a_1$ or $x_2 - y_1 = a_2$, then $y_1 = y_2$, and hence $E(S, -u) = \{y_1\}$, which in turn implies that $E(A, u) \cup \{0\} \subset S - y_1$. If $x_2 - y_1 = a_1$ or $x_1 - y_2 = a_2$, then $x_1 = x_2$, and hence $E(S, u) = \{x_1\}$. In this case $E(A, u) \cup \{0\} \subset x_1 - S$. By Corollary 1, the result follows.

Proof of Corollary 3. The first result follows by Theorem 3, while the second result follows by Theorem 4.

Before giving the proof of Theorem 5, we need a definition and several preliminary results expressed in the form of lemmas.

Definition. If B is compact and u is a unit vector, then the u-convex hull is defined as the smallest u-convex set that contains B, and we denote it by $C_u(B)$.

Lemma A.1. Let B be a compact set and u be a unit vector. Then $C_u(B)$ exists and is given by

$$C_u(B) = \cap\{C{:}B \subset C;\ C \text{ is } u \text{ convex}\}. \qquad \text{(A5)}$$

Proof. It is easy to see that the right-hand side of Eq. (A5) is u convex. It is also true that any u-convex set C, which contains B, contains the right-hand side of Eq. (A5). The result follows.

Lemma A.2. Let B be a compact set and u be a unit vector. Then

$$C_u(B) = \cup\{[x, y]{:}x, y \in B, (x - y) \perp u\}. \qquad \text{(A6)}$$

Proof. Let C_0 be the set on the right-hand side of Eq. (A6). Clearly, if C is any u-convex set containing B, it must contain C_0. Hence, by Lemma A.1, $C_0 \subset C_u(B)$.

To show that $C_u(B) \subset C_0$, it suffices, by Lemma A.1, to prove that C_0 is u convex. Let $b_1, b_2 \in C_0$ be such that $b_1 - b_2 \perp u$. Then by definition there exist $x_1, y_1, x_2, y_2 \in B$ such that $x_1 - y_1 \perp u$, $x_2 - y_2 \perp u$, $b_1 \in [x_1, y_1]$, and $b_2 \in [x_2, y_2]$. Thus there exist $\alpha_1, \alpha_2 \in [0, 1]$ such that $b_j = \alpha_j x_j + (1 - \alpha_j)y_j$ for $j = 1, 2$. By these representations, it is easy to see that $b_j - y_j \perp u$ and $x_j - b_j \perp u$ for $j = 1, 2$. Thus

$$x_1 - y_2 = (x_1 - b_1) + (b_1 - b_2) + (b_2 - y_2) \perp u. \qquad \text{(A7)}$$

A similar argument shows that $x_1 - x_2$, $y_1 - y_2$, and $x_2 - y_1$ are all perpendicular to u. Hence

$$[x_1, y_2], [x_1, x_2], [y_1, y_2], [x_2, y_1] \subset C_0. \qquad \text{(A8)}$$

It is also easy to show that $[b_1, b_2]$ is contained in at least one of the intervals in formula (A8). Thus $[b_1, b_2] \subset C_0$, and hence C_0 is u convex.

Lemma A.3. Let u be a unit vector, A be an autocorrelation support, and S be a support generating A. If A is u convex, then $C_u(S)$ also generates A.

Proof. Since $S - S = A$ and $S \subset C_u(S)$, it suffices to prove that $C_u(S) - C_u(S) \subset A$. By Lemma A.2, it suffices to prove that if $x_1, x_2, y_1, y_2 \in S$ are such that $x_1 - y_1 \perp u$ and $x_2 - y_2 \perp u$, then $[x_1, y_1] - [x_2, y_2] \subset A$. Without loss of generality, we may assume that $[x_1, y_1] - [x_2, y_2] = [x_1 - x_2, y_1 - y_2]$. But $x_1 - x_2, y_1 - y_2 \in A$, and $(x_1 - x_2) - (y_1 - y_2) \perp u$, so this implies that $[x_1 - x_2, y_1 - y_2] \subset A$ by the u convexity of A.

Proof of Theorem 5. Let S be a support generating A, and let $S_1 = C_u(S)$. Then $E(S_1, u)$ and $E(S_1, -u)$ are line segments, which we denote by $[x_1, x_2]$ and $[y_1, y_2]$, respectively, where $a_1 = x_1 - y_1$ and $a_2 = x_2 - y_2$. First we take the case of $|x_1 - x_2| \geq |y_1 - y_2|$. In this case let v be a unit vector perpendicular to u and such that $x_2 = x_1 + \beta v$ and $y_2 = y_1 - \gamma v$, where $\beta \geq \gamma \geq 0$. Then

$$\begin{aligned} m &= \frac{a_1 + a_2}{2} \\ &= \frac{x_1 - y_1 + x_2 - y_2}{2} \\ &= \frac{x_1 - y_1 + x_1 + \beta v - y_1 + \gamma v}{2} \\ &= x_1 + \frac{\beta + \gamma}{2} v - y_1. \end{aligned} \tag{A9}$$

But $(\beta + \gamma)/2 \leq \beta$, and thus $x_1 + [(\beta + \gamma)/2]v \in [x_1, x_2]$, which by Eq. (A9) implies that $m \in S_1 - y_1$. Now let $S_0 = S_1 - y_1$ and note that $\{0, m, a_1\} \subset S_0$. In the case $|y_1 - y_2| \geq |x_1 - x_2|$, let $S_0 = x_1 - S_1$, and a similar argument shows that $\{0, m, a_1\} \subset S_0$.

Now let $\mathcal{S}_0$ be the class of all S_0 generated in the above manner. Then $\mathcal{S}_0$ dominates $\mathcal{S}(A)$ and $\mathcal{S}_0 \subset \mathcal{S}(A)$ by Lemma A.3. Also, $\{0, m, a_1\} \subset \cap\{S_0 : S_0 \in \mathcal{S}_0\}$ and hence, by Theorem 1, $A \cap (A + a_1) \cap (A + m)$ is a single-sided locator set.

An analogous argument shows that $A \cap (A + m) \cap (A + a_2)$ is also a single-sided locator set.

Lemma A.4. Let S be a support, and let A be the autocorrelation support generated by S. Then, for any unit vector u,

$$d(A, u) = 2d(S, u). \tag{A10}$$

Proof. The proof is immediate by Theorem 2.

Proof of Theorem 6. Let S be a support that generates A. By Theorem 2 there exist $x, y \in S$ such that $\{x\} = E(S, u)$, $\{y\} = E(S, -u)$, and $a = x - y$. Since $0 \in A$,

$$E[A \cap (A + a), u] = \{a\}. \tag{A11}$$

By the centrosymmetry of A and Eq. (A11),

$$E[A \cap (A + a), -u] = \{0\}, \tag{A12}$$

and thus

$$d[A \cap (A + a), u] = \frac{d(A, u)}{2} = d(S, u). \tag{A13}$$

As remarked above, in Ref. 7 it was shown that $L_0 = A \cap (A + a)$ is a locator set for A. Thus there exist $x_0, y_0 \in \mathbf{R}^2$ such that $S + x_0 \subset L_0$ and $-S + y_0 \subset L_0$. On combining this with Eqs. (A11) and (A13), we have $x + x_0 = -y + y_0 = a$, which in turn by algebra implies that $x_0 = -y$ and $y_0 = x$. Thus $S - y$ and $-S + x \subset L_0$. Now suppose that both $S - y$ and $-S + x$ are not contained in H. Then there exist $x', y' \in S$ such that

$$\left\langle x' - y - \frac{x - y}{2}, v \right\rangle < -\frac{d}{4} \tag{A14}$$

and

$$\left\langle -y' + x - \frac{x - y}{2}, v \right\rangle < -\frac{d}{4}. \tag{A15}$$

Combining inequalities (A14) and (A15) with the linearity of the inner product, we have

$$\langle y' - x', v \rangle > \frac{d}{2}, \tag{A16}$$

a contradiction to Lemma A.4. Thus either $S - y \subset L$ or $-S + x \subset L$, and we have the desired result.

Proof of Theorem 7. Let S be a support that generates A, and let $a' = x' - y' \in E_l(A, u) \backslash E(A, u)$, where $x', y' \in S$. Since $a' \notin E(A, u)$, either $x' \notin E(S, u)$ or $y' \notin E(S, -u)$ by Theorem 2.

We first consider the case when $x' \notin E(S, u)$. Let $a \in E(A, u)$, and let $a = x - y$, where $x \in E(S, u)$ and $y \in E(S, -u)$. We claim that $y = y'$. In order to show this, assume that $y' \neq y$. Then $(x - y, x - y', x' - y', x' - y) \in \mathcal{P}'$, which implies that $\mathcal{P}' \neq \emptyset$, a contradiction to the hypothesis. Thus $y = y'$, and this implies that $[\{0, a'\} \cup E(A, u)] \subset S - y'$. In the case $y' \notin E(S, -u)$, a similar argument shows that $[\{0, a'\} \cup E(A, u)] \subset x' - S$. Thus, by Corollary 1, we have the desired result.

Before giving the proof of Theorem 8, we need a lemma, which we state and prove.

Lemma A.5. Let u and v be unit vectors such that $u \neq \pm v$, and let B_1 and B_2 be compact sets that are both centered relative to u and v. Also assume that $d(B_1, u) = d(B_2, u)$ and $d(B_1, v) = d(B_2, v)$ and that B_2 dominates B_1. Then either $B_1 \subset B_2$ or $-B_1 \subset B_2$.

Proof. By domination there exist $x_0 \in \mathbf{R}^2$ such that $B_1 + x_0 \subset B_2$ or $-B_1 + x_0 \subset B_2$. We want to show that $x_0 = 0$. Suppose that $x_0 \neq 0$. Then at least one of the four inner products of x_0 with u, v, $-u$, or $-v$ is greater than 0. Suppose that $\langle x_0, u \rangle > 0$. Then

$$\begin{aligned} \sup\{\langle x + x_0, u \rangle : x \in B_1\} &= \langle x_0, u \rangle + \frac{d(B_1, u)}{2} \\ &> \frac{d(B_2, u)}{2} \end{aligned} \tag{A17}$$

and

$$\begin{aligned} \inf\{\langle -x + x_0, -u \rangle : x \in B_1\} &= -\frac{d(B_1, u)}{2} - \langle x_0, u \rangle \\ &< -\frac{d(B_2, u)}{2}. \end{aligned} \tag{A18}$$

Combining inequalities (A17) and (A18) with the centeredness of B_2 implies that $B_1 + x_0 \not\subset B_2$ and $-B_1 + x_0 \not\subset B_2$, a contradiction. Considering the other three possible cases, we are able to derive the same contradiction by similar arguments.

Proof of Theorem 8. Let S be a support that generates A. Without loss of generality, we can assume that S is centered. By Lemma A.4, $d(S, u) = d(L_1, u) = d(L_2, u)$ and $d(S, v) = d(L_1, v) = d(L_2, v)$. When this is combined with Lemma A.5, either $S \subset L_1$ or $-S \subset L_1$. Similarly, either $S \subset L_2$ or $-S \subset L_2$. Thus we have the result in part (a) of Theorem 8 by considering each of the four possible combinations.

The result in part (b) is an immediate consequence of part (a), and, noting that if $A \not\subset L_{12} - L_{12}$, then L_{12} does not dominate S for any support S that generates A. By the proof of part (a), this implies that L_{12}^- dominates S for all supports S generating A.

The result in part (c) of Theorem 8 follows by an argument similar to that given for the proof of part (b).

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