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Reconstruction of a complex-valued object from the modulus of its Fourier transform using a support constraint

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Previously it was shown that one can reconstruct an object from the modulus of its Fourier transform (solve the phase-retrieval problem) by using the iterative Fourier-transform algorithm if one has a nonnegativity constraint and a loose support constraint on the object. In this paper it is shown that it is possible to reconstruct a complexvalued object from the modulus of its Fourier transform if one has a sufficiently strong support constraint. Sufficiently strong support constraints include certain special shapes and separated supports. Reconstruction results are shown, including the effect of tapered edges on the object's support.

INTRODUCTION

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In a number of disciplines, including astronomy, x-ray crystallography, electron microscopy, wave-front sensing, and remote sensing, one encounters the phase-retrieval problem. One wishes to reconstruct f(x, y), an object function, from |F(u, v)|, the modulus of its Fourier transform

$$F(u, v) = |F(u, v)| \exp[i\psi(u, v)] = \mathcal{F}[f(x, y)], \qquad (1)$$

where F denotes Fourier transform. Since the autocorrelation of the object can be computed from the Fourier modulus by $\mathcal{F}^{-1}[[F(u, v)]^2]$, this problem is equivalent to reconstructing an object from its autocorrelation.

For successful reconstruction to be possible, one must have sufficiently strong a priori information (constraints) about the object to make the solution unique. Of course, one has the omnipresent ambiguities that f(x, y), $\exp(i\theta_c)f(x)$ $(-x_0, y - y_0)$, and $\exp(i\theta_c)f^*(-x - x_0, -y - y_0)$, where θ_c is a constant phase, all have the same Fourier modulus. If these omnipresent ambiguities (phase constant, translation, and conjugate image) are the only ambiguities, then we consider the phase-retrieval problem to be unique and do not refer to these ambiguities as ambiguities. For most problems of interest (x-ray crystallography being a notable exception) the object function has finite support. (The support is the set of points over which the object is nonzero.) For the case of two-dimensional (2-D) sampled objects of finite support the solution is almost always unique (when no noise is present).^{1,2} Also, for 2-D continuous objects the solution is probably almost always unique, but the situation is less clear.3,4

The support of the object appears to play the most important role in determining whether the solution is unique. In the one-dimensional (1-D) case the solution is almost always unique if the object is known to have support consisting of at least two disjoint parts satisfying a separation condition,⁵ despite the fact that the 1-D case is usually not unique.⁶ Therefore one would expect the likelihood of ambiguity in the 2-D case to be lower for objects having supports known to have sufficiently separated parts. For some special supports the solution can be shown to be absolutely unique as opposed to almost always unique. Objects including reference points known to satisfy the holography condition are unique. Sampled objects consisting of a rectangular region of support plus a point off one corner of the rectangle (for which one neighboring corner is nonzero) can be shown to be (absolutely) unique.⁷⁻⁹ In addition, sampled objects known to have triangular support (with nonzero corners) and some other shapes including latent reference points are unique.⁹ Recently this has been generalized to show that all sampled objects having supports whose known convex hulls are polygons having no parallel sides are unique.^{10,11} Another class of unique objects includes those consisting of collections of delta functions with separations not satisfying certain redundancy conditions.¹² The library of supports for which the solution is known to be unique is growing as we learn more about this important constraint.

Successful reconstruction requires, in addition to a likelihood of uniqueness, a phase-retrieval algorithm that is not overly sensitive to noise and that converges to a solution using a reasonable amount of computer time. The most widely used phase-retrieval algorithm satisfying these requirements is the iterative Fourier-transform algorithm.¹³⁻¹⁶ A descendant of the Gerchberg-Saxton type of algorithm,^{17,18} it involves the transformation back and forth between the Fourier domain, where the Fourier modulus data are applied, and the object domain, where the a priori object constraints are applied.

For the astronomy problem the only a priori constraint is the object's nonnegativity. Since the autocorrelation of the object can be computed from |F(u, v)|, one also knows the support of the autocorrelation. For extended objects, from the support of the autocorrelation one can usually determine only upper bounds on the support of the object. {}^{12} Therefore the object domain constraints are nonnegativity and a loose support constraint. These constraints have been sufficient to reconstruct a number of computer-simulated astronomical-type objects,^{13,14,19,20} even in the presence of a considerable amount of noise.^{13,15,21}

However, for complex-valued objects the reconstruction

problem is much more difficult. For example, the partially reconstructed images of a complex-valued object shown by Bates and Tan are "barely recognizable and virtually unrecognizable."²² The reason for this lack of success is well known: the loss of the powerful constraint of nonnegativity. However, as will be seen, there are circumstances under which it is possible to reconstruct a complex-valued object.

In this paper we show that the iterative Fourier-transform algorithm is also capable of reconstructing a complex-valued image from its Fourier modulus by using only a support constraint if the support of the object is sufficiently well known (is sufficiently tight), is sufficiently sharp (the object's edges are not tapered too much), and is one of several interesting types.

ITERATIVE FOURIER-TRANSFORM ALGORITHM

For the problem under consideration in this paper it is assumed that |F(u, v)| has been measured and that the support S of f(x, y) is known. The object f(x, y) may be real (nonnegative or bipolar) or may be complex valued (with emphasis here on the latter).

The kth iteration of the iterative Fourier-transform algorithm applied to this problem consists of the following four steps. (1) An input image, $g_k(x, y)$, is Fourier transformed, yielding $G_k(u, v) = |G_k(u, v)| \exp[i\phi_k(u, v)]$; (2) a new Fourierdomain function is formed, using the known Fourier modulus |F(u, v)| with the computed phase: $G_k'(u, v) \ge |F(u, v)| \exp[i\phi_k(u, v)]$; (3) $G_k'(u, v)$ is inverse Fourier transformed to yield $g_k'(x, y)$; (4) a new input is formed by

$$g_{k+1}(x, y) = \begin{cases} g_k'(x, y), & (x, y) \in S \\ g_k(x, y) - \beta g_k'(x, y), & (x, y) \notin S \end{cases}$$
(2)

where β is a constant usually chosen to be anywhere between 0.5 and 1.0 the [performance of the algorithm is not highly sensitive to the choice of the feedback parameter β (Ref. 15)]. Step (4) above embodies the hybrid input-output version of the iterative algorithm, which has been most successful for other phase retrieval problems.¹⁵ The error-reduction version of the iterative algorithm (which most closely follows the Gerchberg-Saxton philosophy) replaces step (4) with

$$g_{k+1}(x, y) = \begin{cases} g_k'(x, y), & (x, y) \in S \\ 0, & (x, y) \notin S \end{cases}$$
(3)

Other versions of the iterative algorithm are applicable as well. $^{\rm 15}$

Progress of the algorithm can be monitored by the objectdomain error metric, a normalized root-mean-squared (nrms) error (the amount by which the output image violates the object-domain constraint):

$$E_{0k} = \left[\frac{\sum_{(x,y) \notin S} |g_{k}'(x,y)|^{2}}{\sum_{(x,y)} |g_{k}'(x,y)|^{2}} \right]^{1/2}.$$
 (4)

When using the error-reduction algorithm it is also appropriate to look at the Fourier-domain error metric

$$E_{Fk} = \left\{ \frac{\sum_{(u,v)} \left[|G_k(u,v)| - |F(u,v)|^2 \right]}{\sum_{(u,v)} |F(u,v)|^2} \right\}^{1/2}.$$
 (5)

The error-reduction algorithm can be proven to converge in the weak sense that $^{15}\,$

$$E_{F(k+1)} \le E_{0k} \le E_{Fk} \le E_{0(k-1)}.$$
(6)

Nevertheless, the error-reduction algorithm is usually much less effective than the hybrid input-output algorithm for which there is no convergence proof.¹⁵ The error-reduction algorithm can be used in conjunction with the hybrid inputoutput algorithm to get a better reading of the residual error during the iterations.¹⁵ Furthermore, when the reconstruction problem is particularly easy, then even the error-reduction algorithm can perform adequately. The error-reduction algorithm is the same as the projection-onto-sets algorithm^{23,24} for this problem; however, the modulus constraint in the Fourier domain is nonconvex, and so the error-reduction algorithm does not enjoy the strong convergence properties of projection onto convex sets.

Note that step (3) of the algorithm can be expressed as

$$G_{k}'(u, v) = G_{k}(u, v)[|F(u, v)|/|G_{k}(u, v)|].$$
(7)

This form can be used instead of the form given in step (2) above, but it obviously has a problem wherever $G_k(u, v) = 0$. This problem has always been easily solved either by using

$$G_{k}'(u,v) = G_{k}(u,v)[|F(u,v)|/(|G_{k}(u,v)| + \delta)],$$
(8)

where δ is a very small positive constant, or by directly setting $G_k'(u,v) = |F(u,v)| \exp[i\phi_k(u,v)]$, where $\phi_k(u,v)$ is the phase of $G_k(u,v)$. This has never caused any difficulties, contrary to speculation.²⁵

RESULTS OF COMPUTER EXPERIMENTS

For all the reconstruction results shown, the reconstruction algorithm employed was the iterative Fourier-transform algorithm, using only a support constraint in the object domain. The initial input to the algorithm was an array of complex random numbers filling the area of the known support. In each case, first 20 iterations of the error-reduction algorithm were performed and then several cycles of iterations were performed, where one cycle of iterations is Kiterations (K = 20 or 40) of the hybrid input-output algorithm with feedback parameter $\beta = 0.7$, followed by 10 iterations of the error-reduction algorithm. Quoted values of E_0 are after error-reduction iterations or after the end of a complete cycle.

Figures 1 and 2 show the results of computer experiments demonstrating the reconstruction of a complex-valued object from the modulus of its Fourier transform by using only a support constraint and exploring the importance of a support constraint having separated parts. The objects for these experiments were generated from a 64×64 portion of a complex-valued SEASAT synthetic aperture radar (SAR) image of an area of land. A binary mask (an array of ones and zeros) was formed to define the desired support constraint. For the first case the support constraint was a pair



Fig. 1. Examples of reconstructing complex-valued objects from the moduli of their Fourier transforms by using a support constraint. (A), (D), (G), and (J): moduli of the complex-valued objects, each having a different support; (B), (E), (H), and (K): the moduli of the images reconstructed by using the iterative Fourier-transform algorithm from the corresponding Fourier moduli, shown in (C), (F), (I), and (L), respectively.

of ellipses separated vertically by a distance greater than the sum of the vertical widths of the ellipses. This separation condition corresponds to the 1-D separation that makes uniqueness likely.⁵ If this separation condition holds, then within the complete autocorrelation of the object the cross correlation of the two ellipses does not overlap the autocorrelations of the individual ellipses. The complex-valued object, the modulus of which is shown in Fig. 1(A), was formed by multiplying the SEASAT SAR image by the binary mask. The modulus of its Fourier transform, shown in Fig. 1(C), was computed from the object imbedded in a $128 \times$ 128 array. This imbedding is done in order to avoid aliasing in the computation of $|F(u, v)|^2$. The modulus of the complex-valued image reconstructed by using the iterative Fourier-transform algorithm with the Fourier modulus and the support constraint is shown in Fig. 1(B). The reconstructed image is essentially perfect, both in modulus and in phase (not shown), up to an additive constant phase. Figure 2, curve 1, shows the object domain nrms error E_0 , given by Eq. (4), as a function of iteration number. The output image looked excellent (visually indistinguishable from the original object) after the second cycle of iterations (120 total iterations). By iteration 820, E_0 bottomed out at 3×10^{-8} ,

presumably limited by roundoff error. In practice, with noise-free data one would ordinarily stop the iterations once E_0 dropped below, say, 0.001.

Figures 1(D), 1(E), and 1(F) show the moduli of the object, the reconstructed image, and the Fourier transform, respectively, for a second case, for which the support is two larger, more closely spaced ellipses not satisfying the separation condition of Ref. 5. Again, the reconstructed image is excellent, but a larger number of iterations was required to achieve an excellent output image (220 iterations, although it was good by 170 iterations) than for the first case. This can be seen from iterations 120 and 170 of Fig. 2, curve 2. E_0 continued to decrease by about a factor of 2 every 100 iterations and was 6×10^{-6} by iteration 1020.

Figures 1(G), 1(H), and 1(I) show the moduli of the object, the partially reconstructed output image after 1020 iterations, and the Fourier transform, respectively, for a third case, for which the support is contiguous, formed by the union of two overlapping ellipses. As seen from E_0 shown in Fig. 2, curve 3, this output image does not agree exactly with the constraints and is not a solution. (Note that for this case E_0 is lower than it was for iteration 120 of the first case, for which the reconstruction was excellent. From case to case E_0 is often poorly correlated with how close the output image is to the original object—it is only a measure of closeness to agreement with the data and constraints.) The algorithm was converging very slowly and was stopped before a solution was found. The quality of the partially reconstructed image is very poor, although it does have some of the features of the object. This type of object appears to be much more difficult to reconstruct than does one with separated support.

Figures 1(J), 1(K), and 1(L) show the moduli of the object, the reconstructed image, and the Fourier transform, respectively, for a fourth case, for which the support is doughnut shaped with an off-center hole. Although the support is contiguous, the number of iterations required for convergence, as shown in Fig. 2, curve 4, is similar to that of the objects having separated support. The output image looked good by iteration 120 and excellent by iteration 170. By iteration 1020, E_0 had decreased to 1.5×10^{-7} . It is interesting that along any 1-D cut through the center of the object, the support does have two separated parts.

Not shown is a fifth case, for which the support was a single ellipse—the larger of the two ellipses shown in Fig. 1(D). As in the third case above, the reconstruction was unsuccessful after several hundred iterations. For this case E_0 is shown in Fig. 2, curve 5.

The results of Figs. 1 and 2 demonstrate that having a support constraint consisting of (at least) two separated parts makes the reconstruction of the object by the iterative Fourier-transform algorithm much easier than reconstruction with only a simple connected support constraint.

The power of the separated support might arise from its holographylike properties. If one of the separated parts is



Fig. 2. Error E_0 as a function of iteration number for reconstruction examples. Curve 1 corresponds to the case of Figs. 1(A), 1(B), and 1(C); curve 2 corresponds to Figs. 1(D), 1(E), and 1(F); curve 3 corresponds to Figs. 1(G), 1(H), and 1(I); curve 4 corresponds to Figs. 1(J), 1(K), and 1(L); curve 5 corresponds to the case of the object being just the larger ellipse in Fig. 1(D).

pointlike and sufficiently separated from the other parts, then it acts as a holographic reference point, and the object can be easily extracted from its autocorrelation.^{26,27} Latent reference points that do not satisfy the holography condition may also be used for some special cases.⁹ One thing that a holographic reference point does is to encode the phase of the Fourier transform in a fringe pattern. For example, the object

$$f(x,y) = A\delta(x - x_0, y) + f_1(x, y)$$
(9)

has Fourier transform

$$F(u, v) = A \exp(-i2\pi u x_0) + F_1(u, v), \tag{10}$$

where $F_1(u, v) = |F_1(u, v)| \exp[i\psi_1(u, v)] = \mathcal{F}[f_1(x, y)]$. Its squared Fourier modulus is

$$|F(u, v)|^{2} = |A \exp(-i2\pi u x_{0}) + F_{1}(u, v)|^{2}$$

= |A|² + |F_{1}(u, v)|² + 2|A||F_{1}(u, v)|\cos[2\pi u x_{0} + \psi_{1}(u, v)]. (11)

For separation x_0 sufficiently large compared with the width of the object, one can see the cos [] fringe in $|F(u, v)|^2$, and the spatial modulation of that fringe by the phase $\psi_1(u, v)$ gives an indication of the phase. If x_0 satisfies the holography condition, then f(x, y) is trivially obtained by spatially filtering the autocorrelation function, which is given by $\mathcal{F}^{-1}[|F(u, v)|^2]$. From this it is seen that the phase $\psi_1(u, v)$ is obtained by taking the phase of $|F(u, v)|^2$ filtered by a single-sided bandpass filter.

For the problem under consideration one does not have holography, since neither of the separated parts $f_0(x, y)$ and $f_1(x, y)$ of

$$f(x, y) = f_0(x, y) + f_1(x, y)$$

is necessarily a delta function, nor is the separation necessarily large enough to satisfy the holography condition. (Furthermore, latent reference points⁹ do not necessarily exist.) Nevertheless, with sufficiently separated parts one still does see a fringelike structure in $|F(u, v)|^2$. As one departs further from the holography condition, both in terms of separation and in the greater extent of the smaller of the two parts, the fringes degrade into a speckle pattern, as seen by comparing Figs. 1(C), 1(F), and 1(I). With a departure from the holography condition, the ability to decipher the phase from the degraded fringes diminishes. From the results shown in Fig. 1 it appears that the iterative Fourier-transform algorithm performs especially well when the Fourier modulus data have any of the fringe structures described above, even when the fringes are substantially degraded.

Note that the supports were chosen to be noncentrosymmetric (except in the fifth case). This was done in order to avoid a potential stagnation problem that can occur for centrosymmetric supports, since in this case $f^*(-x - x_0, -y - y_0)$ is consistent with the support constraint and also has the same Fourier modulus as does f(x, y). As the iterations progress, the partially reconstructed image g'(x, y) may possess features of both f(x, y) and $f^*(-x - x_0, -y - y_0)$. It may be unable to move away from one of those equally valid solutions toward the other, and it may stagnate in this condition. We have developed methods for overcoming this problem,^{16,20} but reconstruction remains easier for noncentrosymmetric supports.



Fig. 3. Examples of reconstructing objects from the moduli of their Fourier transforms by using a triangular support constraint. (A) Object with sharp edges and bright corners, (B) reconstructed image; (C) object with sharp edges and zeroed corners, (D) reconstructed image; (E) object with tapered edges and zeroed corners, (F) partially reconstructed image.

Figure 3 shows the results of computer experiments demonstrating the importance of the sharpness (or tapering) of the edges of an object. Figure 3(A) shows an object having triangular support and nonzero values in its three corners. These conditions ensure that the object is unique among objects having that support, and under these conditions there is a closed-form recursive algorithm for reconstructing it.⁹ For the present experiment, the image, shown in Fig. 3(B), was reconstructed by using the iterative Fourier-transform algorithm (rather than the recursive algorithm). Nonnegativity was not used as a constraint, although the object happens to be nonegative. In this case the algorithm converged rapidly to the solution.

Another example of the ability to reconstruct an object with this type of support constraint by using the iterative Fourier-transform algorithm is shown in Ref. 28. There, the object was a pure-phase wave front transmitted through a triangular aperture.

Since the recursive reconstruction algorithm and the uniqueness proof require the three corners to be nonzero, we wanted to determine the importance of nonzero corners to the iterative Fourier-transform reconstruction algorithm. The same experiment was performed for the object shown in Fig. 3(C), which is identical to the object shown in Fig. 3(A) but with the corners zeroed out. The support constraint used in the iterative algorithm was the same triangular support as was used for the case above. The reconstructed image, shown in Fig. 3(D), is the correct solution, but convergence was slower in this case than for the case of the object having three bright corners. Therefore the brightness of the corners has an effect on convergence but is not crucial as far as the iterative algorithm is concerned (it is crucial to the success of the recursive algorithm).

The effect of the sharpness of the edges of the object was also investigated. A third object, having tapered edges, shown in Fig. 3(E), was formed by multiplying the object shown in Fig. 3(C) by a tapering function along each of its three edges. Thus this third object has the same triangular support as the other two cases described above, but it has small values near the edges of the support. An attempt was made to reconstruct the image from its Fourier modulus by using the iterative Fourier-transform algorithm with the triangular support constraint. The output image resulting after several hundred iterations is shown in Fig. 3(F). Although the image is easily recognizable, it has a noisy appearance. It does not represent a solution, since it is not in perfect agreement with the data and constraints. The algorithm was stagnating, and the iterations were halted before a solution was found. This example shows that if one does not use a nonnegativity constraint, then the sharpness of the edges of the object is very important to the ability of the iterative algorithm to reconstruct an image by using only a support constraint in the object domain.

CONCLUSIONS

Previously it was shown that by using the iterative Fouriertransform algorithm one could reconstruct a nonnegative object from the modulus of its Fourier transform with a loose support constraint. In general the reconstruction of com-

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plex-valued objects is considerably more difficult than for real-valued, nonnegative objects.²² The results shown here demonstrate the possibility of reconstructing complex-valued objects if one has a tight enough support constraint that is one of a number of special types of support constraints. These special types of support constraints include supports having separated parts and supports for which the object can be reconstructed by using the recursive algorithm with latent reference points; the latter class of objects includes objects with supports whose convex hulls have no parallel sides. One would expect to be able to find other supports as well for which the iterative Fourier-transform algorithm performs successfully. Simple symmetric support constraints such as single ellipses (circles) or rectangles do not work well. The algorithm also works much better for objects having sharp edges than for objects having tapered edges. Further research is being performed to examine in more detail the effects of the shape of the support, the amount of edge tapering, and the presence of noise on the ability to reconstruct a complex-valued object from the modulus of its Fourier transform by using only a support constraint. Portions of this work were reported in Refs. 20, 29, and 30.

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