Reconstruction of objects having latent reference points

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A simple recursive algorithm is proposed for reconstructing certain classes of two-dimensional objects from their autocorrelation functions (or equivalently from the modulus of their Fourier transforms—the phase-retrieval problem). The solution is shown to be unique in some cases. The objects contain reference points not satisfying the holography condition but satisfying weaker conditions. Included are objects described by Fiddy *et al.* [Opt. Lett. **8**, 96 (1983)] satisfying Eisenstein's thorem.

INTRODUCTION

In a number of disciplines, including astronomy, x-ray crystallography, electron microscopy, and wave-front sensing, one encounters the phase-retrieval problem. One wishes to reconstruct $f(m, n)$, an object function, from $|F(p, q)|$, the modulus of its Fourier transform, where

$$\begin{aligned} F(p,q) &= |F(p,q)|\exp[i\psi(p,q)] = \mathcal{F}[f(m,n)] \\ &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)\exp[-i2\pi(mp/M + nq/N)], \end{aligned} \tag{1}$$

where $m, p = 0, 1, \ldots, M-1$ and $n, q = 0, 1, \ldots, N-1$. The discrete transform is employed here since in practice one deals with sampled data in a computer. The problem of reconstructing the object from its Fourier modulus is equivalent to reconstructing the Fourier phase, $\psi(p, q)$, from the Fourier modulus; since once one has the phase as well as the modulus, one can easily compute $f(m, n)$ by the inverse (discrete) Fourier transform. $r_f(m, n)$, the (aperiodic) autocorrelation of $f(m, n)$, is given by[1]

$$r_f(m,n) = \sum_{j=0}^{M-1}\sum_{k=0}^{N-1} f(j,k)f^*(j-m, k-n) \tag{2}$$

$$= \mathcal{F}^{-1}[|F(p,q)|^2], \tag{3}$$

where the asterisk denotes complex conjugate. Note that the autocorrelation is Hermitian: $r_f(-m, -n) = r_f^*(m, n)$. Note also that in order to avoid aliasing during the computation of $|F(p, q)|^2$, it is necessary to have $f(m, n) = 0$ for $M/2 \leq m \leq M-1$ and for $N/2 \leq n \leq N-1$; this will be assumed throughout this paper. Then there is no difference between the periodic (cyclic) and aperiodic autocorrelation. (For x-ray crystallography this is usually not the case, and the results of this paper do not apply.) Since the autocorrelation function is easily computed from the Fourier modulus by Eq. (3), the phase-retrieval problem is equivalent to reconstructing an object from its autocorrelation function.

Several phase-retrieval algorithms have been proposed, all of them requiring some additional measurements or constraints on the solution. Examples include a reference point at least one object diameter from the object[2] (giving rise to the holography condition[3]), a second intensity measurement in another plane[4,5] (in electron microscopy or wave-front sensing), nonnegativity and limited spatial extent[6–8] (in astronomy), atomic models[9] (in x-ray crystallography), and objects consisting of collections of points having nonredundant spacings.[10]

Here it is pertinent to review the case of holography. Suppose that $f(m, n)$ consists of an object of interest, $g(m, n)$, plus an unresolved (delta-function-like) point, referred to as the reference point, i.e.,

$$f(m,n) = A\delta(m-m_0, n-n_0) + g(m,n), \tag{4}$$

where $\delta(m, n)$ is a two-dimensional (2-D) Kronecker delta function. Then the autocorrelation can be written as the sum of four terms,

$$\begin{aligned} r_f(m,n) = |A|^2\delta(m,n) + r_g(m,n) + Ag^*(m_0 - m, n_0 - n) \\ + A^*g(m+m_0, n+n_0), \end{aligned} \tag{5}$$

the final term of which is the cross-correlation of the reference point with the object of interest and is simply proportional to a translate of the object of interest. If the distance from the reference point to the object of interest exceeds the diameter of the object of interest, then the fourth term in Eq. (5) is nonoverlapping with the other terms, and the object of interest is reconstructed by simple inspection of the autocorrelation. Then the holography condition is satisfied.[2,3] If the amplitude and position of the reference point are unknown (except that the reference point satisfies the holography condition), then the object can be reconstructed only to within a complex factor A^* and to within a translation, and there would be a twofold ambiguity as to whether the object is given by the fourth term or the third term (the conjugate image) of Eq. (5).

In this paper we describe an algorithm for reconstructing certain objects having reference points that do not satisfy the holography condition. For these cases the reference points may be referred to as latent reference points, because they do not immediately yield the object as would a holographic reference point; rather, a degree of development is required before their usefulness emerges.

In Section 2 the question of the uniqueness of the solution is reviewed. In Section 3 the new reconstruction algorithm is described as it is applied to three different classes of objects. Additional comments on the reconstruction algorithm are included in Section 4.

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2. UNIQUENESS OF THE SOLUTION

When one measures only the Fourier modulus, then the uniqueness of the solution is a central question. One of course always has the twofold (180° rotated or conjugate image) ambiguity since $|\mathcal{F}[f(m,n)]| = |\mathcal{F}[f^*(-m,-n)]|$; and translations of $f(m,n)$ and the multiplication of $f(m,n)$ by a constant phase factor $\exp(i\theta)$ (where θ is a real constant) also have no effect on $|F(p,q)|$. If these are the only ambiguities, then we consider the solution of the phase-retrieval problem to be unique.

Bruck and Sodin[11] considered objects consisting of a rectangular grid of delta functions having various complex amplitudes (or equivalently, a 2-D sequence), which have Fourier transforms that can be expressed as polynomials. These are the types of objects assumed by Eqs. (1) and (2), and we refer to such objects as sampled objects. They showed that, for sampled objects, a lack of uniqueness of the solution to the phase-retrieval problem is equivalent to the factorability of the polynomial, and therefore one-dimensional (1-D) objects of length L have a 2^{L-1}-fold ambiguity.[11] This result corresponds to the analogous theory for 1-D continuous functions.[12] On the other hand, polynomials of two (or more) variables are known to be only rarely factorable (i.e., they are usually irreducible). Consequently, for 2-D sampled objects the solution to the phase-retrieval problem is usually unique. An analogous theory for 2-D continuous functions is not yet available.

Uniqueness Condition Due to Eisenstein's Theorem

Although most 2-D sampled objects are, as discussed above, uniquely related to the modulus of their Fourier transforms, it is of interest to know conditions that ensure uniqueness. Such a condition was recently put forward by Fiddy *et al.*[13] They considered the class of sampled objects whose support is contained in the union of a rectangle and an isolated point (A) below and to the right of the rectangle, as shown in Fig. 1(a). By way of example, the rectangular region in Fig. 1(a) contains five columns and four rows of points. The object must also be nonzero both at point A and at point B in the lower left corner of the rectangle. Points A and B are referred to as the reference points, and they do not satisfy the holography condition. If these conditions are satisfied, then the Fourier transform of the object satisfies Eisenstein's theorem, making it an irreducible 2-D polynomial and guaranteeing that the solution to the phase retrieval problem is unique. They demonstrated the power of these conditions by reconstruction experiments using the input–output iterative Fourier-transform algorithm.[6,7] First, they performed a reconstruction experiment on the Fourier modulus of a particular object that did not have a reference point A. After 250 iterations, a poor reconstruction resulted. But when a new object was formed by adding a reference point A off its corner making it satisfy the conditions, then a good reconstruction was obtained after only 20 iterations.[13] Note that this does not prove that the original object (without the point A) was nonunique: the failure of the iterative reconstruction algorithm may only be an indication of local minima in the error function. In fact, when the reference point A had a small value, a poor reconstruction was obtained in spite of the fact that irreducibility (and uniqueness) was ensured. Only when a large value for A was used did the reconstruction become easier.[13] Apparently the use of a large enough value for A also ensures that there are no local minima.

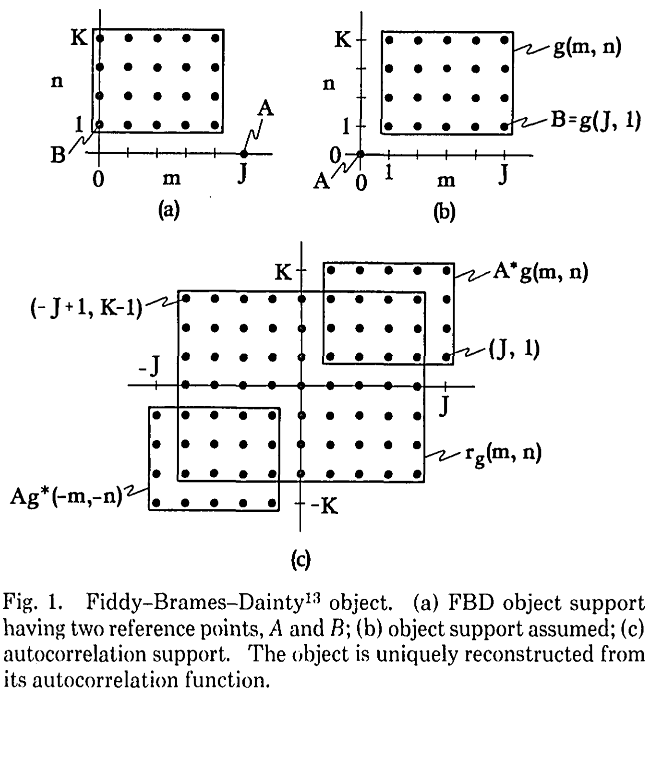

Fig. 1. Fiddy–Brames–Dainty[13] object. (a) FBD object support having two reference points, A and B; (b) object support assumed; (c) autocorrelation support. The object is uniquely reconstructed from its autocorrelation function.

3. NEW RECONSTRUCTION ALGORITHM

For certain classes of sampled objects having reference points not satisfying the holography condition, we present a new reconstruction algorithm having a fixed number of steps. This new algorithm is related to the Dallas[5] recursive algorithm for phase retrieval from two intensity measurements but requiring only a single intensity measurement (the Fourier modulus) and solving the equations in a certain order such that the problem of a growing tree of solutions[5] is avoided. First the algorithm will be described for the type of object described above, and later for a wider class of objects.

A. Fiddy–Brames–Dainty Objects

For mathematical simplicity, consider a sampled object whose support is contained in the regions shown in Fig. 1(b). Its uniqueness properties are the same as those of the objects considered in Fig. 1(a) since the supports are mirror images of one another. The object can be expressed as in Eq. (4) with $m_0 = n_0 = 0$:

$$f(m,n) = A\delta(m,n) + g(m,n),$$

where $g(m,n)$ is that part of $f(m,n)$ contained in the rectangular region of support, and $A = f(0,0) \neq 0$. In this case, $g(m,n)$ is zero outside $1 \leq m \leq J$ and $1 \leq n \leq K$; and it is assumed that $f(J,1) = g(J,1) = B \neq 0$, and $g(m,K) \neq 0$ for at least one value of m. We will refer to objects satisfying these constraints as Fiddy–Brames–Dainty (FBD) objects having FBD regions of support.

The autocorrelation, $r_f(m,n)$, of $f(m,n)$ is given by the four terms of Eq. (5) with $m_0 = n_0 = 0$, the supports of which are contained in the sets of points illustrated in Fig. 1(c). From this figure, it can be clearly seen that the rightmost column and the uppermost row of $r_f(m,n)$ are simply equal to $A^*g(m,n)$:

$$r_f(J, n) = A^* g(J, n), \qquad n = 1, \ldots, K, \tag{6}$$

$$r_f(m, K) = A^* g(m, K), \qquad m = 1, \ldots, J. \tag{7}$$

Therefore, for $m = J$ and for $n = K$, one can reconstruct $g(m, n)$ to within a constant factor A^* by simple inspection of $r_f(m, n)$. In effect, the holography condition is in force for the row and column opposite reference point A, and that row and that column are reconstructed by using reference point A.

The value of A can be obtained as follows: From Eq. (2), it is seen that there is only one nonzero term in the summation for the upper left corner point in the autocorrelation:

$$r_f(-J+1, K-1) = g(1, K) g^*(J, 1) = B^* g(1, K). \tag{8}$$

Also, from Eqs. (6) and (7),

$$r_f(J, 1) = A^* g(J, 1) = A^* B, \tag{9}$$

$$r_f(1, K) = A^* g(1, K). \tag{10}$$

Combining Eqs. (8)–(10) yields, assuming that $r_f(-J+1, K-1) \neq 0$,

$$|A|^2 = \frac{r_f(J, 1) r_f^*(1, K)}{r_f^*(-J+1, K-1)}. \tag{11}$$

Since without loss of generality we can arbitrarily fix the phase of any one point in $f(m, n)$, we set the phase of A equal to zero; A is then given unambiguously by the positive square root of Eq. (11). If $r_f(-J+1, K-1) = 0$, then one can obtain a similar expression for $|A|^2$ using the first nonzero point, $r_f(m, K-1)$, to the right of $r_f(-J+1, K-1)$. Since A is known, $g(J, n)$ and $g(m, K)$ can be determined unambiguously from Eqs. (6) and (7). Note that $B = g(J, 1) = r_f(J, 1)/A^*$.

Having the values of the top row and rightmost column of $g(m, n)$, one can then solve for the leftmost column in the second step of the algorithm. From Eq. (2), the point of the autocorrelation just below $r_f(-J+1, K-1)$ has only two nonzero terms,

$$r_f(-J+1, K-2) = g(1, K) g^*(J, 2) + g(1, K-1) g^*(J, 1). \tag{12}$$

Solving,

$$g(1, K-1) = [r_f(-J+1, K-2) - g(1, K) g^*(J, 2)]/B^*, \tag{13}$$

where $g(J, 1) = B$. Since all the quantities of the right-hand side of Eq. (13) are known and $B \neq 0$, one can unambiguously compute $g(1, K-1)$. Similarly, the next lower point in the autocorrelation is given by

$$r_f(-J+1, K-3) = g(1, K) g^*(J, 3) + g(1, K-1) g^*(J, 2) + g(1, K-2) g^*(J, 1). \tag{14}$$

Since all the quantities in this linear equation are known except for $g(1, K-2)$, and since $g(J, 1) \neq 0$, one can solve unambiguously for $g(1, K-2)$. In a similar fashion, one can recursively solve for all the values $g(1, n)$ (the first column on the left) using the values of $r_f(-J+1, n-1)$ in this second step of the reconstruction. In a sense the column $m = 1$ was solved using the latent reference point B, which required the solution of column $m = J$ before it could become effective.

Having the first column on the left and the first column on the right of $g(m, n)$, one can then solve for the second column on the right in the third step, using A as the latent reference point. From Eq. (2), the points of the autocorrelation in column $(J-1)$ are given by

$$r_f(J-1, n) = g(J-1, n) A^* + \sum_{k=n+1}^{K} g(J, k) g^*(1, k-n), \tag{15}$$

for $n = 1, \ldots, K-1$. Since, for any n, $g(J-1, n)$ is the only unknown in Eq. (15), and since $A \neq 0$, $g(J-1, n)$ is uniquely determined from Eq. (15). Thus the values of $g(m, n)$ in column $(J-1)$ are reconstructed using the values in column $(J-1)$ of the autocorrelation.

The reconstruction algorithm continues in the manner described above. In the fourth step, one can recursively solve for $g(2, n)$ using the latent reference point B and the values of $r_f(-J+2, n-1)$, $n = K-1, K-2, \ldots, 2, 1$. In the fifth step, one can solve for $g(J-2, n)$ using the latent reference point A and the values of $r_f(J-2, n)$, $n = 1, \ldots, K-1$. One continues the procedure until all the columns of $g(m, n)$ are reconstructed, giving a complete and unambiguous reconstruction of $g(m, n)$, and therefore of $f(m, n)$.

If $g(1, K) \neq 0$, then one can alternatively use that point as B and perform the reconstruction as described above, but reversing the roles of the rows and columns.

It was recently noted that Eisenstein's theorem allows for the rectangular region of support (see Fig. 1) to be extended over (in the same column as) point A. However, in that case, there is no simple recursive algorithm for reconstructing the object.

B. Support Uniqueness for Fiddy-Brames-Dainty Objects

In the reconstruction method described above, it was implicitly assumed that the support of the object function was known. However, as will be shown by what follows, such an assumption is not necessary, since an FBD object can be shown to be an FBD object from its autocorrelation. In order to use theorems[10] relating to reconstructing the support of an object from the support of its autocorrelation function, during the discussion of the object and autocorrelation supports we assume that the object function is real and nonnegative. (It might happen that what follows may, with appropriate modifications, also be true for complex-valued objects; but this would require further development.)

Given only the support of the autocorrelation, one can usually only put an upper bound on the support of the object.[10] Such upper bounds, sets that can contain translates of the supports of all possible solutions, we refer to as locator sets. One such locator set is the intersection of the autocorrelation support with a translate of itself, where the translate is such that the center of the second autocorrelation support is within the first autocorrelation support.[10] Assuming that $r_f(-J+1, K-1) \neq 0$, and translating the one autocorrelation support so that it is centered at $(-J+1, K-1)$, one arrives at the locator set shown in Fig. 2 for the case of the FBD object support shown in Fig. 1(b). In addition, since the autocorrelation is $2J+1$ pixels wide and $2K+1$ pixels high, the object must be $J+1$ pixels wide and $K+1$ pixels high. Since the object support must be contained within the locator set shown in Fig. 2, which is $J+2$ pixels wide and $K+2$ pixels high, the object support must include either the lower left point or the upper right point but not both. Keeping either one of these

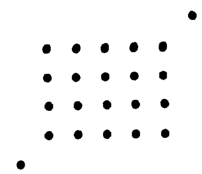

Fig. 2. Locator set containing all possible solutions, used to show that the support solution is unique.

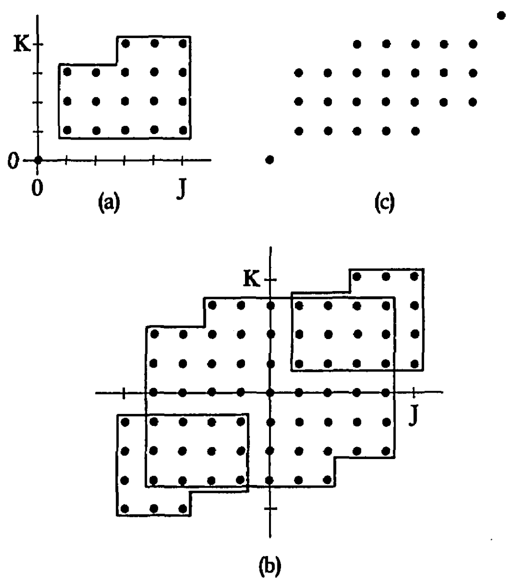

Fig. 3. Alternative case. (a) Object support; (b) autocorrelation support; (c) locator set.

two points and discarding the other, one is left with the support of the object (or the 180° rotated version—the twofold ambiguity). Suppose, on the other hand, that $r_f(-J+1, K-1) = 0$. For example, suppose that the object support is that shown in Fig. 3(a). Then the autocorrelation support is that shown in Fig. 3(b). A locator set, formed by taking the intersection of this autocorrelation support with one translated to be centered at the first nonzero point in row $(K-1)$, is shown in Fig. 3(c). As in the case of Figs. 1 and 2, since the autocorrelation is $2K+1$ pixels high, the object must be $K+1$ pixels high, and therefore either the lower right or the upper left point (but not both) in Fig. 3(c) must be within the object support. Suppose we take the lower left point as being within the object (choosing the upper right point will result in the 180° rotated solution). Then, since the autocorrelation is $2J+1$ pixels wide and therefore the object must be $J+1$ pixels wide, the object must be contained within the first $J+1$ columns on the left of Fig. 3(c), which is just the support of the object as shown in Fig. 3(a). From these examples it can be seen that, in general, if the object is an FBD object, then its support can be reconstructed from the autocorrelation function, from which it is also evident that the object has an FBD support.

C. Triangular Objects

Other types of objects, in addition to FBD objects, can be reconstructed by the recursive method. In this and the next section the reconstruction of two other classes of objects are shown. Consider, for example, objects whose support is contained in the triangular shape shown in Fig. 4(a). Assuming that the object's support is known *a priori*, it has been shown that for this particular object shape the boundaries can be reconstructed in a simple way,[14] assuming $A, B, C \neq 0$. Since the vector spacings between points A and B, B and C, and C and A are each unique, from the corner points in the autocorrelation, as shown in Fig. 4(b), we have

$$r(0,K) = f(0,K)f^*(0,0) = CA^*, \tag{16a}$$

$$r(J,-K) = f(J,0)f^*(0,K) = BC^*, \tag{16b}$$

$$r(J,0) = f(J,0)f^*(0,0) = BA^*. \tag{16c}$$

Combining these gives

$$|A|^2 = \frac{r^*(0,K)r(J,0)}{r(J,-K)}. \tag{17}$$

Without loss of generality the phase of A can be chosen to be zero, and then A is given by the positive square root of Eq. (17). Then we can also compute

$$B = r(J,0)/A^*, \tag{18a}$$

$$C = r(0,K)/A^*. \tag{18b}$$

Then the values of the leftmost column of the object are given by

$$f(0,n) = r(-J,n)/B^*, \tag{19}$$

the values of the bottom row are given by

$$f(m,0) = r(m,-K)/C^*, \tag{20}$$

and the values of the diagonal are given by

$$f(m,K-m) = r(m,K-m)/A^*. \tag{21}$$

From this point one could determine the remainder of the object by solving systems of equations,[5,14] but an easier way

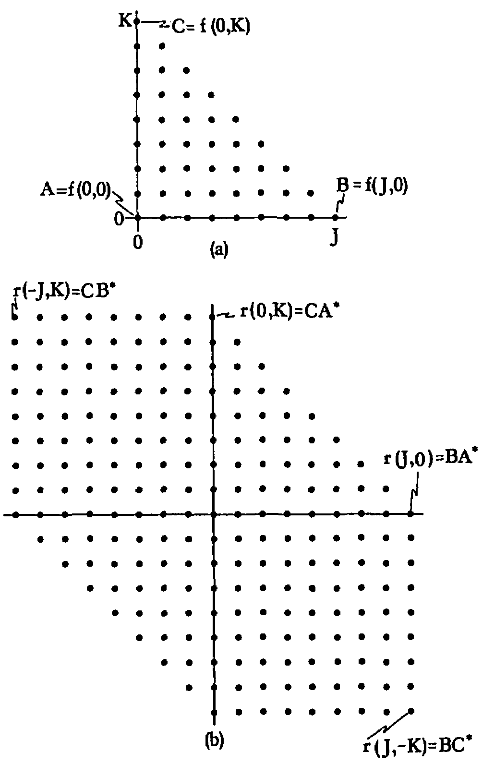

Fig. 4. Triangular-shaped object. (a) Object support; (b) autocorrelation support. The object is uniquely (among triangular-shaped solutions) reconstructed from its autocorrelation function.

$$\begin{array}{ccc} 1 & & \\ 2 & 2 & \\ 1 & 2 & 1 \end{array} \qquad \begin{array}{ccc} 1 & 1 & \\ 1 & 3 & 1 \\ & 1 & 1 \end{array}$$

(a) (b)

$$\begin{array}{ccccc} 1 & 2 & 1 & & \\ 2 & 8 & 8 & 2 & \\ 1 & 8 & 15 & 8 & 1 \\ & 2 & 8 & 8 & 2 \\ & & 1 & 2 & 1 \end{array} \qquad \begin{array}{cc} 1 & \\ 1 & 1 \end{array}$$

(c) (d)

Fig. 5. Specific triangular-shaped object. (a) The object; (b) a second nontriangular-shaped solution; (c) the common autocorrelation function; (d) the function used to synthesize objects shown in (a) and (b).

is possible if one cleverly chooses the order in which the equations are solved. In particular, only one linear equation with one unknown at a time need be solved, and the solution at each step is unique, if one solves in the following order. In a similar manner as was done for the FBD objects, solve for the points in column $m = 1$ using B as a latent reference point, and solve for the points in row $n = 1$ using C as a latent reference point. Next solve for the points in column $m = 2$ using B as a latent reference point, and solve for the points in row $n = 2$ using C as a latent reference point. This procedure is continued until all of $f(m, n)$ is reconstructed. Other orderings for the recursive solution of the equations are also possible.

The solution given above for the triangular-shaped object is unique among objects having that support but may not be unique among all objects. Momentarily restricting $f(m, n)$ to the case of nonnegative objects, one can use the autocorrelation support tri-intersection reconstruction for convex sets[10] to show that there exists a family of object supports that have autocorrelation supports equal to the one shown in Fig. 4(b). One member of that family is the original object support shown in Fig. 4(a). Another member is an object support resembling the autocorrelation support shown in Fig. 4(b) but only half its size. For these latter members there is no simple recursive reconstruction algorithm as there is for the triangular-shaped object.

Further insights can be obtained by analyzing a simple case. A case for which there are exactly two different solutions (not counting 180°-rotated versions) can be obtained by starting with nonsymmetric functions $h_1(x, y)$ and $h_2(x, y)$ whose Fourier transforms are nonfactorable and generating a first object, which is $h_1(x,y)$ convolved with $h_2(x,y)$, and a second object, which is $h_1(x,y)$ convolved with $h_2(-x,-y)$ (i.e., the cross correlation).[15] Two such objects, their common autocorrelation function, and the $h_1(x,y) = h_2(x,y)$ used to generate them are shown in Figs. 5(a) through 5(d), respectively. In this case one obtains the "unique" solution shown in Fig. 5(a) if triangular support is assumed, and the "unique" solution shown in Fig. 5(b) if the only other possible support is assumed.

Since relatively few 2-D objects have factorable Fourier transforms, the ambiguous example shown in Fig. 5 is unusual. If one started with a random object having the same support as the object in Fig. 5(b), and if one incorrectly assumed that the object had the same triangular support as the object in Fig. 5(a), then one would obtain what at first glance would appear to be a triangular-shaped solution. In the process of calculating the solution one would use only the points on the perimeter of the autocorrelation function, with which the "solution" would be consistent. However, on further inspection one would usually find that the triangular-shaped solution is inconsistent with the interior points of the autocorrelation function. Only in the unlikely event that the original object's Fourier transform is factorable would the triangular-shaped solution be completely consistent with the autocorrelation function. Therefore if the given autocorrelation function admits to a possible solution by the recursive method, then one should reconstruct the solution with the assumed support, then compute its autocorrelation function and compare it with the given autocorrelation function to determine whether the assumed support is valid.

D. Another Case

For a final example, consider objects contained within the support shown in Fig. 6(a). Comparing it with Fig. 1(b), it would be a FBD object if it were not for the fact that $B = 0$. Assuming that the support of the object is known, it can be reconstructed by the following recursive steps if points A and $B' \neq 0$ and if either point C' or $C'' \neq 0$. First $f(J, 2), \ldots, f(J, K)$ and $f(2, K), \ldots, f(J-1, K)$ are solved using A as the reference point. A can be determined from an equation similar to Eqs. (11) and (17). Next $C' = f(1, K-1)$, then $f(1, K-2)$, $\ldots$, then $f(1, 2)$ are solved using B' as the latent reference point. Next $f(J-1, 1)$ is solved using C' or C'' as the latent reference point. Next $f(1, 1)$ is solved using B' as the latent reference point. Next $f(J-1, 2), \ldots, f(J-1, K-1)$ are solved using A as the latent reference point. Then the pattern repeats: solve for $f(2, K-1), \ldots, f(2, 2)$ recursively using B', then solve for $f(J-2, 1)$ using C' or C'', then solve for $f(2, 1)$

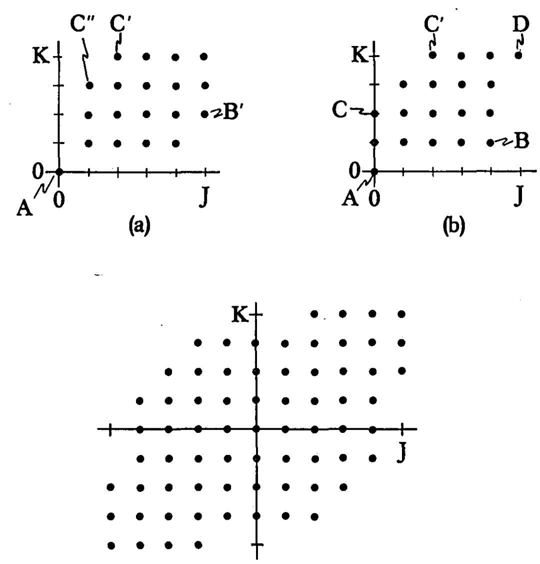

Fig. 6. Another case related to FBD objects. (a) Object support; (b) alternative support reconstruction; (c) autocorrelation support. The object is reconstructed from its autocorrelation function, with two solutions.

using B', then solve for $f(J-2, 2), \ldots, f(J-2, K-1)$ using A, etc., until all the columns are solved.

The solution for this object is unique among objects having support contained within the support shown in Fig. 6(a). However, another support may also be possible. In a manner similar to that used in connection with Figs. 1–3, the possible support solutions can be narrowed down to those of Fig. 6(a) and Fig. 6(b), given the autocorrelation support shown in Fig. 6(c). For the support shown in Fig. 6(b) one can reconstruct the object unambiguously by solving a proper sequence of equations using latent reference points A, B, C, C', and D. Therefore, given the autocorrelation function whose support is shown in Fig. 6(c), at most two (and more probably only one) solutions are possible, and each can be reconstructed using a simple recursive algorithm depending on the support shown in either Fig. 6(a) or 6(b).

4. CONCLUSIONS

A simple recursive algorithm has been devised for reconstructing an object from its autocorrelation function (or its Fourier modulus). It works for several types of sampled objects having latent reference points, including those satisfying the conditions described by FBD. The manner in which the algorithm results in a unique solution constitutes a proof of uniqueness for FBD objects (but not necessarily for all objects whose Fourier transforms satisfy Eisenstein's theorem). One of the principal lessons learned here is that the detailed shape of the boundary of an object plays a crucial role in determining the uniqueness of the solution to the phase-retrieval problem.

One might be able to use this method for continuous objects (as opposed to inherently sampled objects) if a dense enough sampling of the autocorrelation is available.[5]

Since the algorithm involves repeatedly taking differences and dividing by the values of the latent reference points, it may be sensitive to noise and may require latent reference points having large values for an accurate reconstruction. (This may be related to the fact that a large value of A was required for a successful reconstruction using the iterative Fourier-transform algorithm.[13]) Not all the (nonsymmetric) points in the autocorrelation are used by this algorithm; improved accuracy should be expected if the reconstruction algorithm were modified to use also those additional points. Those additional points may also be used to distinguish whether assumptions about the support of the object (when more than one support solution is possible) are valid. For the best results one should finish the reconstruction by using the output of this reconstruction method as the initial input to the iterative Fourier-transform algorithm,[6,7] which finds a solution that is most consistent with both the measured data and the *a priori* constraints.

The reconstruction algorithm proposed here is applicable to only a relatively small number of types of objects. However, the approach of carefully selecting the order in which the equations are solved should be helpful in the more general use of Dallas's method by limiting the growth of the tree of solutions.[5]

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