Ambiguity of phase retrieval for functions with disconnected support

T. R. Crimmins and J. R. Fienup

Radar and Optics Division, Environmental Research Institute of Michigan, P.O. Box 8618, Ann Arbor, Michigan 48107

Received February 2, 1981

Questions are raised concerning the uniqueness of solutions to the phase-retrieval problem for functions with disconnected support. A counterexample is presented showing the importance of considering the flipping of infinite proper subsets of nonreal zeros.

INTRODUCTION

In this Letter, questions are raised concerning some claims by Greenaway[1] and Bates.[2] Their papers are concerned with the question of uniqueness of solutions to the phase-retrieval problem. This problem, in the one-dimensional case, can be stated as follows.

Let f be a complex-valued function on the real line that vanishes outside some finite interval. Let F be its Fourier transform. Given the modulus of F on the real line, i.e., $|F(u)|$ for all real u, the problem is to reconstruct the original function f from this information. The general uniqueness question is: *How many other functions, $g \neq f$, exist that vanish outside some finite interval and whose Fourier transforms satisfy $|G(u)| = |F(u)|$ for all real u?*

GREENAWAY'S PAPER

Greenaway[1] considers a situation in which the unknown function f is known to be zero outside the union of two disjoint intervals (a,c) and (d,b). In other words,

$$f = g + h,$$

where g is zero outside (a,c) and h is zero outside (d,b) (see Fig. 1).

Now let F, G, and H be the Fourier transforms of f, g, and h, respectively, extended by analyticity into the complex plane:

$$F(w) = \int_{-\infty}^{\infty} f(x)e^{-iwx}dx,$$

where $w = u + iv$ and u and v are real. The modulus of F on the real line, i.e., $|F(u)|$, is given.

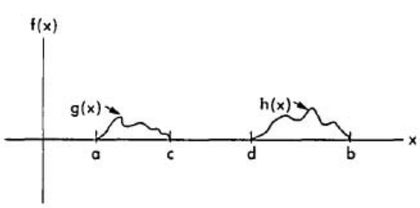

Fig. 1. Member of the class of functions with disconnected support. Note: Although the functions g and h are represented here as positive real functions, they can be complex valued.

The question is: *To what extent do the conditions described above determine the function f?*

The functions

$$e^{i\alpha}f(x + \beta) \quad \text{and} \quad e^{i\alpha}\overline{f(-x + \beta)},$$

where the overbar denotes complex conjugation and α and β are real, have the same Fourier modulus on the real line as does f. If any of these functions are also zero outside the union of the intervals (a,c) and (d,b), then they satisfy all the requirements and qualify as alternative solutions. These solutions will be said to be *associated* with the solution f.

Now the revised question is: *Are there any other solutions* not *associated with f?*

Let w_0 be a nonreal zero of F, and let

$$F_1(w) = F(w)\,\frac{w - \overline{w_0}}{w - w_0}.$$

The function F_1 can be viewed as being gotten from F by first removing a zero at w_0 and then adding a zero at $\overline{w_0}$. In other words, the zero at w_0 has been "flipped" about the real line. Now for real w, $w = u$,

$$\left|\frac{u - \overline{w_0}}{u - w_0}\right| = 1,$$

and therefore

$$|F_1(u)| = |F(u)| \qquad \text{for all real } u.$$

Hofstetter[3] and Walther[4] proved that if f_1 is any function that vanishes outside some finite interval and $|F_1(u)| = |F(u)|$ for all real u, then F_1 is gotten from F by flipping various sets of nonreal zeros of F and multiplying by a constant of modulus 1 and by an exponential function. In particular, if F_1 is obtained from F by flipping the set of all its nonreal zeros, then its inverse transform f_1 satisfies

$$f_1(x) = \overline{f(-x)},$$

and thus, if f_1 vanishes outside the union of (a,c) and (d,b), then f_1 is a solution associated with f. (Here, if a zero of F has multiplicity n, it must be flipped n times.)

Now let $Z(F)$ denote the set of all nonreal zeros of F.

Greenaway claims that if F_1 is obtained from F by flipping any *proper* subset S of $Z(F)$ [i.e., $S \neq Z(F)$] and if f_1 vanishes outside the union of (a,c) and (d,b), then all the points in S are zeros of both G and H.

0030-3941/81/081026-03$00.50

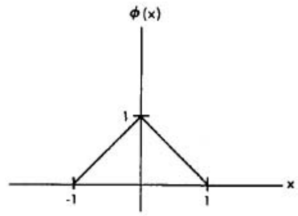

Fig. 2. The function $\phi(x)$.

Thus, if G and H have no zeros in common (which would usually be the case if g and h are gotten more or less randomly from the real world), it would follow that the only solutions are f and its associated solutions. Bates[5] also considers this problem from another point of view.

Greenaway's claim is true in the special case in which F has only a finite number of nonreal zeros. (Actually, Greenaway's proof holds only for the more restricted case in which F has a finite number of nonreal zeros of order 1. However, the case of higher-order zeros can be taken care of by an extension of his argument. See Ref. 6.)

The following counterexample shows that Greenaway's claim is not true in general. In this counterexample, the set $Z(F)$ is infinite and S is an infinite proper subset of $Z(F)$.

Counterexample

Let

$$\phi(x) = \begin{cases} 1 - |x| & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}.$$

(See Fig. 2.) Then the Fourier transform Φ of ϕ is given by

$$\Phi(w) = \frac{\sin^2(w/2)}{(w/2)^2} = \operatorname{sinc}^2(w/2).$$

Note that Φ has no nonreal zeros.

Now let

$$g(x) = 8\phi(x)$$

and

$$h(x) = 2\phi(x-4) + 3\phi(x-8) + 2\phi(x-12).$$

Then $G = 8\Phi$ has no nonreal zeros, and hence G and H have no nonreal zeros in common. Let

$$\begin{aligned} f(x) &= g(x) + h(x) \\ &= 8\phi(x) + 2\phi(x-4) + 3\phi(x-8) + 2\phi(x-12) \end{aligned}$$

and let

$$a = -1, \quad c = 1, \quad d = 3, \quad b = 13.$$

(See Fig. 3.) Then $a < c < d < b$, the intervals (a,c) and (d,b) are disjoint, and f is zero outside the union of (a,c) and (d,b). The Fourier transform of f is

$$\begin{aligned} F(w) &= (8 + 2e^{-4iw} + 3e^{-8iw} + 2e^{-12iw})\Phi(w) \\ &= 2(e^{-4iw} + 2)(e^{-8iw} - 0.5e^{-4iw} + 2)\Phi(w). \end{aligned} \tag{1}$$

Now let

$$g_1(x) = 4\phi(x)$$

and

$$h_1(x) = 7\phi(x-4) + 4\phi(x-12).$$

Then $G_1 = 4\Phi$ has no nonreal zeros, and hence G_1 and H_1 have no nonreal zeros in common. Let

$$\begin{aligned} f_1(x) &= g_1(x) + h_1(x) \\ &= 4\phi(x) + 7\phi(x-4) + 4\phi(x-12). \end{aligned}$$

(See Fig. 3.) Then f_1 is also zero outside the union of (a,c) and (d,b). The Fourier transform of f_1 is

$$\begin{aligned} F_1(w) &= (4 + 7e^{-4iw} + 4e^{-12iw})\Phi(w) \\ &= 2(2e^{-4iw} + 1)(e^{-8iw} - 0.5e^{-4iw} + 2)\Phi(w) \\ &= 2e^{-4iw}\,(e^{4iw} + 2)(e^{-8iw} - 0.5e^{-4iw} + 2)\Phi(w). \end{aligned} \tag{2}$$

It follows from Eqs. (1) and (2) that

$$F_1(w) = e^{-4iw}\left(\frac{e^{4iw} + 2}{e^{-4iw} + 2}\right) F(w).$$

Now, for real w, $w = u$,

$$\left| e^{-4iu}\,\frac{e^{4iu} + 2}{e^{-4iu} + 2} \right| = 1.$$

Therefore

$$|F_1(u)| = |F(u)| \qquad \text{for all real } u.$$

Thus f and f_1 are both solutions, and it is clear that they are not associated.

In order to see which zeros must be flipped to get F_1 from F, let

$$\Gamma_1(w) = e^{-4iw} + 2$$

and

$$\Gamma_2(w) = e^{-8iw} - 0.5e^{-4iw} + 2.$$

Then

$$F(w) = 2\Gamma_1(w)\Gamma_2(w)\Phi(w) \tag{3}$$

and

$$F_1(w) = 2e^{-4iw}\,\overline{\Gamma_1(\overline{w})}\Gamma_2(w)\Phi(w). \tag{4}$$

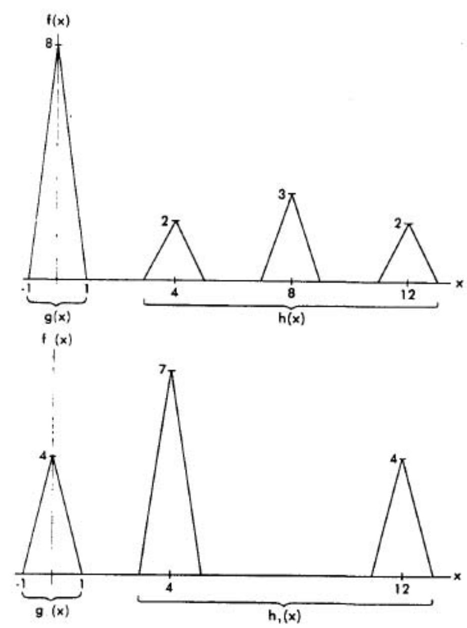

Fig. 3. Functions $f(x)$ (above) and $f_1(x)$ (below) have the same Fourier modulus.

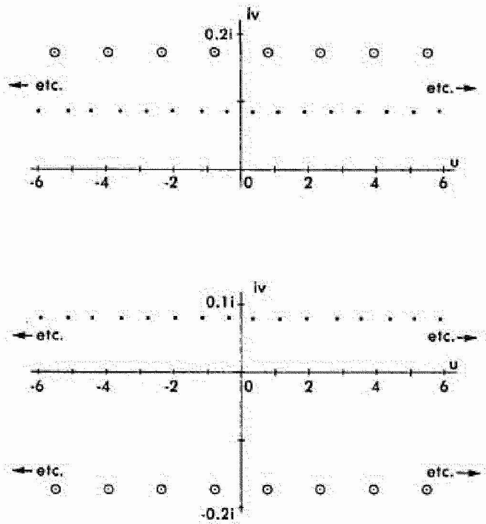

Fig. 4. Above: nonreal zeros of F. Below: nonreal zeros of F_1. The circled zeros are flipped.

Since Φ has no nonreal zeros and e^{-4iw} is never zero, it follows from Eqs. (3) and (4) that

$$Z(F) = Z(\Gamma_1) \cup Z(\Gamma_2)$$

and

$$Z(F_1) = \overline{Z(\Gamma_1)} \cup Z(\Gamma_2),$$

where

$$\overline{Z(\Gamma_1)} = \{\overline{w}:\quad w \in Z(\Gamma_1)\}.$$

Thus the zeros of F that are in $S = Z(\Gamma_1)$ are flipped. The sets $Z(\Gamma_1)$ and $Z(\Gamma_2)$ are given by

$$Z(\Gamma_1) = \left\{\frac{\pi}{4} + \frac{\pi}{2}n + \frac{i}{4}\log 2:\quad n = 0, \pm 1, \pm 2, \ldots\right\}$$

and

$$Z(\Gamma_2) = \left\{\pm\frac{1}{4}\tan^{-1}\sqrt{31} + \frac{\pi}{2}n + i\frac{1}{8}\log 2:\quad n = 0, \pm 1, \pm 2, \ldots\right\}.$$

(See Fig. 4.) The flipping of the zeros in S is followed by multiplication by the exponential e^{-4iw}. The latter simply has the effect of translating f_1 into the proper position.

In the above example the function ϕ could be replaced by any function that is zero outside the interval $(-1,1)$ and whose Fourier transform has no nonreal zeros. For example, ϕ could be replaced by

$$\phi_1(x) = (\phi * \phi)(2x),$$

where $*$ denotes convolution, or by

$$\phi_2(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}.$$

BATES'S PAPER

Bates[2] considers the situation in which

$$f(x) = \sum_{n=1}^{N} f_n(x),$$

where each f_n is zero outside an interval I_n and the intervals I_n, $n = 1 \ldots N$, are pairwise disjoint. He claims that if the Fourier transforms F_n have no nonreal zeros common to all of them, then f and its associated solutions are the only functions with compact support and whose Fourier transforms have the same moduli as that of the Fourier transform of f. Thus Bates claims even more than Greenaway does. Therefore the above example is also a counterexample to Bates's claim.

It should be noted that the function f to which Bates is applying this argument is the one-dimensional projection of a function defined on the plane. He concludes from this argument that, in the two-dimensional case, solutions with disconnected support are almost always unique (up to associated solutions). This conclusion regarding two-dimensional uniqueness may well be true for other reasons discussed by Bruck and Sodin.[7]

COMMENTS

We stress that functions with disconnected support whose Fourier transforms have only a finite number of nonreal zeros form a special class of functions. It can be shown (see Ref. 6) that such functions satisfy certain special conditions. Thus it is quite common for functions with disconnected support gotten more or less randomly from the real world to have Fourier transforms with an infinite number of nonreal zeros. The fact that one is able, in practice, to compute only a finite number of them does not change these conclusions. One cannot claim to be making statements about the uniqueness of a solution unless the entire infinity of nonreal zeros of its Fourier transform is properly considered.

Finally, we note that the example presented here does not imply that the attempt to reconstruct functions from the moduli of their Fourier transforms is hopeless. It can be shown[6] that a stronger separation condition on the separate parts of the support of f (i.e., that the parts are separated by certain greater intervals than assumed by Greenaway and Bates) does imply that f and its associated solutions are the only solutions.

This research was supported by the U.S. Air Force Office of Scientific Research under contract no. F49620-80-C-0006.

REFERENCES

1. A. H. Greenaway, "Proposal for phase recovery from a single intensity distribution," Opt. Lett. **1**, 10–12 (1977).
2. R. H. T. Bates, "Fringe visibility intensities may uniquely define brightness distributions," Astron. Astrophys. **70**, L27–L29 (1978).
3. E. M. Hofstetter, "Construction of time-limited functions with specified autocorrelation functions," IEEE Trans. Inf. Theory **IT-10**, 119–126 (1964).
4. A. Walther, "The question of phase retrieval in optics," Opt. Acta **10**, 41–49 (1963).
5. R. H. T. Bates *et al.*, "Self-consistent deconvolution. I: Theory," Optik (Stuttgart) **44**, 183–201 (1976).
6. T. R. Crimmins and J. R. Fienup, "Phase retrieval for functions with disconnected support," in Appendix B of J. R. Fienup, "High resolution imaging of space objects," interim scientific report to Air Force Office of Scientific Research, ERIM Report No. 145400-7-P (Environmental Research Institute of Michigan, Ann Arbor, Mich., March 1981).
7. Yu. M. Bruck and L. G. Sodin, "On the ambiguity of the image reconstruction problem," Opt. Commun. **30**, 304–308 (1979).