TABLE II
(a) WORST CASE AND AVERAGE MAGNITUDE ERRORS ON TRAINING DATA USING THE ESTIMATION FORMULA. (b) WORST CASE AND AVERAGE MAGNITUDE ERRORS ON TRAINING DATA USING A CONSTANT MINIMUM RMS ERROR ESTIMATE (c) WORST CASE AND AVERAGE MAGNITUDE ERRORS ON TRAINING DATA USING A CONSTANT MINIMUM WORST CASE ERROR ESTIMATE.

	Data File	Description	Worst Case (Hz)	Average Magnitude (Hz)
(a)	Set 1	TI data–men, women, and children	148	98
	Set 2	TI data–adult males	142	98
	Set 3	Peter Ladefoged data–adult males	87	58
	Set 4	Barney-Peterson data–adult males	123	64
(b)	Set 1	TI data–men, women, and children	586	221
	Set 2	TI data–adult males	494	129
	Set 3	Peter Ladefoged data–adult males	373	115
	Set 4	Barney-Peterson data–adult males	539	146
(c)	Set 1	TI data–men, women, and children	487	250
	Set 2	TI data–adult males	390	191
	Set 3	Peter Ladefoged data–adult males	320	133
	Set 4	Barney-Peterson data–adult males	385	239

TABLE III
WORST CASE ERRORS ON TEST FILES

Test Data File	Data Files on Which Parameters were Optimized			
	Set 1	Set 2	Set 3	Set 4
Set 1	148	425	620	577
Set 2	415	142	257	191
Set 3	457	266	87	170
Set 4	524	234	191	123

TABLE IV
AVERAGE MAGNITUDE ERRORS ON TEST FILES

Test Data File	Data Files on Which Parameters were Optimized			
	Set 1	Set 2	Set 3	Set 4
Set 1	98	284	373	372
Set 2	287	98	257	78
Set 3	230	141	58	62
Set 4	267	103	100	64

considered as exceptions. Two of the 256 cells on the graphics surface were dedicated to these two sounds, the (x, y) coordinates being selected based on their respective first and second formant values. A board was mounted to the graphics pen with a lever and four switches which could be operated by the thumb and four fingers, respectively.

The first pass of the hardware fixed the third and fourth formant frequencies at 2500 and 3500 Hz, respectively. A later version estimated these frequencies based on the product formula described above and produced more natural sounding speech, particularly the sounds *IY* as in "see" and *UW* as in "food."

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Comments on "The Reconstruction of a Multidimensional Sequence from the Phase or Magnitude of Its Fourier Transform"

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***Abstract*—When one imposes a nonnegativity constraint, one usually can reconstruct a two-dimensional sequence of finite support from the modulus of its Fourier transform using an iterative algorithm, even when the initial estimate is an array of random numbers.**

In a recent paper,[1] the description of an iterative algorithm for reconstructing a sequence from the magnitude of its Fourier transform unintentionally gives the appearance of discussing an algorithm published earlier [1]. In the following,

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[1] M. H. Hayes, *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-30, pp. 140–154, Apr. 1982.

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the differences between the algorithm and experiments described by Hayes[1] and those published earlier [1] are clarified.

Hayes[1] reviews both the problem of reconstructing a sequence from the phase of its Fourier transform and the problem of reconstructing a sequence from the magnitude of its Fourier transform. For the latter problem, he describes an iterative algorithm for solving the problem as follows. "Specifically, this algorithm involves the repeated Fourier transformation between the time and frequency domains where, in each domain, the known information about the desired sequence is imposed on the current estimate. In the time domain, for example, a sequence is constrained to have a given region of support whereas in the frequency domain, the sequence is constrained to have a given transform magnitude."[1] He then shows examples where the algorithm described above fails. This failure should not reflect poorly on the earlier work [1] since the algorithm described in the quotation above and the experiments performed by Hayes differ in important ways from the earlier work. In Hayes experiments, both the type of information which was assumed to be known and the reconstruction algorithm which was used differed from those of the earlier work [1].

Hayes is correct in stating[1] that the magnitude of the Fourier transform is insufficient to uniquely specify a sequence; additional information or constraints are required. Depending on the application, one often has available additional information or constraints, and a reconstruction may then be possible [2], [3]. Two important constraints which often occur (as in astronomy) are a known support (or bounds on the support) of a sequence, and the constraint that the sequence be nonnegative [4]. Unlike the algorithm used by Hayes,[1] the iterative method described earlier [1] primarily uses the *nonnegativity* constraint. Using the iterative algorithm, we have been very successful in reconstructing two-dimensional nonnegative sequences from their Fourier magnitude [1]-[6]. In this case, the sequences must have finite support, but it is possible to reconstruct them even when the support is not known. Except for special cases, it is not possible to determine the support of a sequence from the support of its autocorrelation (which is the inverse Fourier transform of the squared Fourier magnitude) [7], so the support information is usually not available anyway. One can only place upper bounds on the support [7]. If an upper bound on the support is utilized during the iterations, then the algorithm converges faster (in about 100 or 200 iterations for our work) than when using only the nonnegativity constraint (in which case we found that several hundred iterations are required).

Unlike the algorithm used by Hayes, the iterative algorithm described earlier [1] does not simply satisfy the constraints (nonnegativity and bounds on the support) in the time-domain step of the iteration. Such an algorithm, which we refer to as the error-reduction algorithm, was discussed earlier [1] where it is noted that, "For the present application, the error-reduction approach requires an impractically large number of iterations for convergence." It is only a version of the input-output algorithm [1]-[6] which is capable of converging in 100 or so iterations.

Hayes found that "... if the initial estimate used in the iteration has a Fourier transform with the correct magnitude and either zero phase or random phase, then the iteration will not generally converge to the correct sequence."[1] However, using the input-output algorithm with a nonnegativity constraint, we obtained good reconstruction results when the algorithm was initialized with arrays of random numbers [1]-[6]. The algorithm has also been shown to be surprisingly insensitive to noise [5].

When the error-reduction algorithm was used with a nonnegativity constraint (as well as a support constraint), it took *many thousands* of iterations for convergence [3], [6]. Therefore, if one were to employ the error-reduction algorithm without a nonnegativity constraint, then one would expect convergence to take much longer, if it ever converges. Consequently, it is consistent with our experience that the type of reconstruction experiments performed by Hayes would be unsuccessful.

Of course, there are situations for which the nonnegativity constraint does not apply. Then one might wonder whether it is possible to reconstruct a sequence of finite support from its Fourier magnitude. Theory ([8], Hayes[1]) seems to indicate that the solution will usually be unique. However, as shown by Hayes, the error-reduction algorithm is not a practical approach to finding the solution. One might possibly succeed using an accelerated algorithm, such as the input-output algorithm or a gradient search method [6], but this is an area that needs further work.

It should also be noted that in the phase retrieval problem of X-ray crystallography, one reconstructs the three-dimensional electron density function from its Fourier magnitude. For that problem, one has the constraints that the electron density is nonnegative and that it consists of a discrete number of atoms. For that problem, a number of reconstruction methods have been developed [9]. For the phase retrieval problem in electron microscopy, for which both the wave function and its Fourier transform are complex valued, one has the additional constraint of knowing the magnitude of the wave function. For that problem, the error-reduction algorithm has been shown to perform very well [10], [11].

In conclusion, Hayes' remark that "... even for those sequences which are uniquely defined by their magnitude, it appears that a practical algorithm is yet to be developed for reconstructing a sequence from only its magnitude"[1] is strictly true when no other information is available; however, for a number of important applications, there is auxiliary information, such as a nonnegativity constraint, and practical reconstruction algorithms do exist.

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