Conservation law for electromagnetic fields in a space-time-varying medium and its implications

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We discuss the propagation of electromagnetic waves in a space-time modulated medium, for which neither energy nor momentum remains conserved. We have found a conservation law when modulation moves at a constant speed v in a traveling-wave fashion. We use this law to study how light is scattered inside such a medium. We consider both the subluminal ($v < c$) and superluminal ($v > c$) cases and find that they differ considerably in terms of what remains conserved. We show that the total number of photons is conserved in the subluminal case. In contrast, it is the difference between the forward- and backward-moving photons that remains conserved in the superluminal case. We also study the reciprocity issue for a space-time modulated medium and find that reciprocity holds in all situations. We develop a quantum formulation of our scattering problem and show that the subluminal case is similar to the action of a beam splitter. In the superluminal case, photon pairs can be generated from vacuum in a fashion analogous to the two-mode frequency down-conversion process.

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I. INTRODUCTION

Considerable attention has focused recently on wave propagation in a medium whose refractive index varies with time [1–3]. Since external modulation of the medium breaks the time-translation symmetry, the usual conservation law of energy does not hold for such a medium, which makes it possible to observe new phenomena. Nonconservation of energy is allowed in the discussion of open systems that does not include energy used to modulate the medium in which electromagnetic waves are propagating.

Two types of temporal modulation are possible in practice. In one kind, changes in refractive index happen spatially everywhere at the same time. If a sudden index change occurs, known as a temporal boundary, an incident wave can break into two waves, traveling in opposite directions. This phenomenon is usually called temporal reflection and refraction [4–6]. The frequency of the reflected wave is shifted from that of the incident wave, and energy of the electromagnetic wave is also different, which is a consequence of the time-translational symmetry being broken. As another example, when a periodic temporal modulation is applied to a medium, a photonic time crystal is formed, which supports modes that undergo exponential amplification [7–10].

The second type of modulation, called space-time modulation, has also attracted attention. In this case, the medium's refractive index changes in both space and time. In this work, we are interested in a type of modulation that takes the form of a traveling wave. Mathematically, the medium is characterized by the permittivity and permeability that vary as $\epsilon(z - vt)$ and $\mu(z - vt)$, where v is the velocity of modulation and can have any value without conflicting with the theory of relativity [2]. This type of modulation can be realized in experiments by sending an intense pump pulse through a nonlinear medium. Many interesting phenomena have been explored, such as the formation of a space-time photonic crystal [11], synthetic Fresnel drag [12], and luminal amplification [13]. The quantum-mechanical description of a moving grating, especially in the transluminal case, was also discussed in Refs. [14,15].

The question we ask is whether a conservation law exists in the case of space-time modulation. For a purely static medium ($v = 0$), electromagnetic energy is conserved as a consequence of time-translational symmetry. In the case of purely temporal modulation ($v \to \infty$), the medium exhibits space-translation symmetry, leading to the conservation of momentum [16]. For any other value of v, neither the energy nor the momentum is conserved since both symmetries are broken simultaneously. However, a mixed space-time translational symmetry still occurs, and one would expect some conserved quantity to exist.

In this work, we employ Maxwell's equations with a matrix technique to derive a conservation law. Further, we analyze the implications of this conservation law for a scattering problem, where an incident pulse scatters off a traveling-wave modulation. We study how our conservation law differs in the subluminal case ($v < c$) and superluminal ($v > c$) case. In the former case, the total number of photons is conserved before and after the scattering process. In contrast, it is the difference in the number of photons traveling in the forward and backward directions that is conserved in the superluminal case. Photon-number conservation has been discussed by Pendry [17,18]. However, he focused on a periodic modulation scheme relevant to photonic space-time crystals. Here, we make no such assumption. We also address the issue of reciprocity by considering incident waves incident on the opposite sides of the medium. Furthermore, we provide a quantum-mechanical formulation of the problem and show that scattering from subluminal modulation acts like the incident wave has gone through a beam splitter. In contrast, scattering from superluminal modulation exhibits evolution similar to a frequency down-conversion process in which photons are generated in pairs (one going forward and the other going backward). Using our formulation, one can study the propagation of nonclassical light in a space-time modulated medium.

II. CONSERVED QUANTITY

We focus on a medium whose parameters are modulated in a traveling-wave fashion: $\epsilon = \epsilon(z - vt)$ and $\mu = \mu(z - vt)$, where ϵ and μ are the permittivity and permeability, respectively. Both of them are assumed to be real and positive as we do not consider dissipation. A linearly polarized plane wave is traveling in the z direction inside such a medium. Assuming that the electric field points in the x direction and magnetic field in the y direction, we use the notation $\mathbf{E} = \hat{e}_x E$ and $\mathbf{H} = \hat{e}_y H$. Maxwell's curl equations then take the form

$$\begin{aligned} \frac{\partial E}{\partial z} &= -\frac{\partial}{\partial t}[\mu(z - vt)H], \\ \frac{\partial H}{\partial z} &= -\frac{\partial}{\partial t}[\epsilon(z - vt)E]. \end{aligned} \tag{1}$$

These equations assume that modulation of parameters occurs through a phenomenon such as the Pockels or the Kerr effect, but the medium itself is not moving. For a physically moving medium, they would not have the form given in Eqs. (1) as a consequence of relativity.

We perform a Galilean transform of coordinates using

$$z' = z - vt, \quad t' = t. \tag{2}$$

In this moving frame, Eqs. (1) become

$$\begin{aligned} \frac{\partial}{\partial t'}[\epsilon(z')E] &= -\frac{\partial H}{\partial z'} + v\frac{\partial}{\partial z'}[\epsilon(z')E], \\ \frac{\partial}{\partial t'}[\mu(z')H] &= -\frac{\partial E}{\partial z'} + v\frac{\partial}{\partial z'}[\mu(z')H]. \end{aligned} \tag{3}$$

To find the normal modes in the moving frame, we multiply both sides of Eqs. (3) with i and allow E and H to be complex. Then, we can write them in a matrix form as

$$i\frac{\partial}{\partial t'}\begin{pmatrix} \epsilon & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} 0 & \hat{p}_{z'} \\ \hat{p}_{z'} & 0 \end{pmatrix}\begin{pmatrix} 1 & -v\mu \\ -v\epsilon & 1 \end{pmatrix}\begin{pmatrix} E \\ H \end{pmatrix}, \tag{4}$$

where we have defined the Hermitian operator $\hat{p}_{z'} = -i\frac{\partial}{\partial z'}$, similar to the momentum operator in quantum mechanics.

The preceding equation resembles a Schrödinger equation and allows us to use Dirac's notation to represent an electromagnetic field. We define a wave function as

$$|\phi\rangle = \begin{pmatrix} 1 & -v\mu \\ -v\epsilon & 1 \end{pmatrix}\begin{pmatrix} E \\ H \end{pmatrix}, \tag{5}$$

and rewrite Eq. (4) in the following form:

$$i\frac{\partial}{\partial t'}M|\phi\rangle = \begin{pmatrix} 0 & \hat{p}_{z'} \\ \hat{p}_{z'} & 0 \end{pmatrix}|\phi\rangle, \tag{6}$$

where the Hermitian matrix M is given by

$$M = \begin{pmatrix} \epsilon & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} 1 & -v\mu \\ -v\epsilon & 1 \end{pmatrix}^{-1} = \frac{1}{1 - v^2\epsilon\mu}\begin{pmatrix} \epsilon & v\epsilon\mu \\ v\epsilon\mu & \mu \end{pmatrix}. \tag{7}$$

At this point, we need to differentiate between the subluminal and superluminal cases. We will first consider the subluminal case to find the form of the conserved quantity. Then, we can easily show that this quantity is conserved in both the subluminal and superluminal cases. In the subluminal case, $|v| < 1/\sqrt{\epsilon\mu}$ at all points in space, the matrix M is positive definite, and we can use $M = A^2$, where A is also a Hermitian matrix. The use of matrix A allows us to rewrite Eq. (6) in the form

$$i\frac{\partial}{\partial t'}A|\phi\rangle = A^{-1}\begin{pmatrix} 0 & \hat{p}_{z'} \\ \hat{p}_{z'} & 0 \end{pmatrix}A^{-1}A|\phi\rangle. \tag{8}$$

Introducing $|\psi\rangle = A|\phi\rangle$, we write this equation in the form of a standard Schrödinger equation as (with $\hbar = 1$)

$$i\frac{\partial}{\partial t'}|\psi\rangle = H|\psi\rangle, \tag{9}$$

where the Hermitian operator is defined as

$$H = A^{-1}\begin{pmatrix} 0 & \hat{p}_{z'} \\ \hat{p}_{z'} & 0 \end{pmatrix}A^{-1}. \tag{10}$$

Following a standard procedure in quantum mechanics, we can find "energy eigenstates" of Eq. (9), which correspond to the normal modes of the system, in the form

$$H|\psi_i\rangle = \omega_i|\psi_i\rangle, \tag{11}$$

where ω_i is the mode frequency for the ith normal mode. The orthonormal condition suggests that

$$\langle\psi_j|\psi_i\rangle = \langle\phi_j|A^2|\phi_i\rangle \propto \delta_{ij}, \tag{12}$$

with $M = A^2$. Since H is Hermitian, the norm of the state $\langle\psi|\psi\rangle = \langle\phi|A^2|\phi\rangle$ should be conserved during the evolution. To see how it is related to the energy and momentum associated with an electromagnetic field, we recall that the inner product of $|\psi\rangle = \binom{f}{g}$ is defined as

$$\langle\psi_1|\psi_2\rangle = \int_{-\infty}^{\infty}(f_1^* \quad g_1^*)\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}dz'. \tag{13}$$

Using this definition and Eq. (7), we obtain

$$\begin{aligned} \langle\phi|M|\phi\rangle &= \int_{-\infty}^{\infty}(E^* \quad H^*)\begin{pmatrix} 1 & -v\epsilon \\ -v\mu & 1 \end{pmatrix}\begin{pmatrix} \epsilon & 0 \\ 0 & \mu \end{pmatrix} \\ &\quad \times \begin{pmatrix} 1 & -v\mu \\ -v\epsilon & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & -v\mu \\ -v\epsilon & 1 \end{pmatrix}\begin{pmatrix} E \\ H \end{pmatrix}dz' \\ &= \int(E^* \quad H^*)\begin{pmatrix} 1 & -v\epsilon \\ -v\mu & 1 \end{pmatrix}\begin{pmatrix} \epsilon & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} E \\ H \end{pmatrix}dz' \\ &= \int \epsilon|E|^2 + \mu|H|^2 - v\epsilon\mu(E^*H + EH^*)dz'. \end{aligned} \tag{14}$$

This is the conserved quantity that we were looking for.

We can use Eq. (14) to define an effective Hamiltonian density as

$$\mathcal{H} = \tfrac{1}{2}[(\epsilon|E|^2 + \mu|H|^2) - v\epsilon\mu(E^*H + EH^*)], \quad (15)$$

where the factor of $1/2$ was added to ensure that it reduces to the conventional energy density in the limit $v \to 0$. The total effective Hamiltonian is conserved. Directly taking the time derivative of $\mathcal{H}$ and using Eq. (3), we obtain

$$\frac{\partial \mathcal{H}}{\partial t'} = -\frac{\partial}{\partial z'}\left[\frac{1}{2}(1 + v^2\epsilon\mu)(E^*H + EH^*) - v(\epsilon|E|^2 + \mu|H|^2)\right]. \quad (16)$$

This relation is a generalization of Poynting's theorem and leads to fundamental constraints on the reflection and transmission coefficients in a scattering problem. Equation (16) implies that the spatial integral of the Hamiltonian density is invariant with respect to time.

It is worth stressing that the conserved quantity in Eq. (14) actually holds for any modulation velocity v. This is so because Eq. (16) can be derived by directly differentiating Eq. (15) and using Eq. (3) for any modulation velocity v. Our discussion focused on the subliminal case to obtain the functional form of the conserved quantity, but the resulting conservation law holds in all cases. By direct inspection of the conserved quantity, it turns out to be a linear combination of the energy and momentum of the electromagnetic wave. This is reasonable since the system possesses a mixed space-time translational symmetry.

We can convert Eq. (16) back to the laboratory frame. After some algebra, we obtain

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial}{\partial z}\left[\frac{1}{2}(E^*H + EH^*) - \frac{v}{2}(\epsilon|E|^2 + \mu|H|^2)\right]. \quad (17)$$

Let us define the generalized Poynting vector as

$$\mathcal{S} = \frac{1}{2}(E^*H + EH^*) - \frac{v}{2}(\epsilon|E|^2 + \mu|H|^2). \quad (18)$$

For a pulsed beam, we expect the field quantities to vanish as t or z goes to infinity. This leads to the following two conserved quantities:

$$\frac{d}{dt}\left(\int \mathcal{H}dz\right) = 0, \quad \frac{d}{dz}\left(\int \mathcal{S}dt\right) = 0. \quad (19)$$

III. PHOTON-NUMBER CONSERVATION

We are interested in using the conservation law that we derived to study the scattering of an incident light pulse with a moving modulation. As mentioned earlier, we need to consider the subluminal and superluminal cases separately. Figure 1 shows schematically how an incident light pulse scatters from a moving modulated region in these two cases, assuming that the region is finite in both space and time and the medium is homogeneous on its two sides. We allow the medium's parameters to be different on the two sides of this region. In practice, this may be achieved by sending a long pump pulse to modulate the medium, then the green part corresponds to the rising edge of the pump pulse (where the index is changing) while the dark blue region would be inside the long pump pulse (where the index is basically a constant). In the subluminal case, the speed of modulation is slower than the speed of light c in all regions (including inside the modulated region).

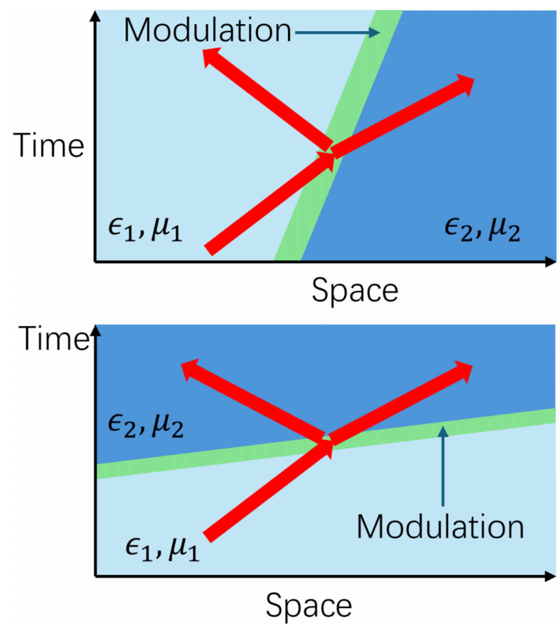

FIG. 1. Schematic of an incident light pulse scattering from a modulation band (green) in the subluminal (top) and superluminal cases.

In the superluminal case, the speed of modulation is faster than c everywhere. We do not consider the transluminal case [13], where the modulation is moving faster than the c in some regions and slower elsewhere. Comparing the two cases shown in Fig. 1, we notice that reflected and transmitted waves exist on two different sides of the modulated region in the subluminal case, but they exist on the same side in the superluminal case. This is a consequence of causality, and it results in different behaviors in the two cases.

A. Subluminal case

Let us first discuss the subluminal case and consider incident light in the form of a pulse, finite in both space and time. As shown in Fig. 1, the pulse is propagating in a homogeneous medium characterized by ϵ_1 and μ_1 until it arrives in the modulated region. Thus, its electric and magnetic fields can be written in the form

$$E(z,t) = A_i(z - c_1 t), \quad H(z,t) = \tfrac{1}{\eta_1}A_i(z - c_1 t), \quad (20)$$

where A_i is the amplitude of the incident pulse, $c_1 = (\epsilon_1\mu_1)^{-1/2}$ is the speed of light in this region, and $\eta_1 = \sqrt{\mu_1/\epsilon_1}$ is the impedance of the medium. We evaluate the integral of Hamiltonian density defined in Eq. (15), which should be invariant under time evolution:

$$\int \mathcal{H}dz = \epsilon_1\left(1 - \frac{v}{c_1}\right)\int |A_i(z - c_1 t)|^2 dz. \quad (21)$$

As seen in the laboratory frame, the incident pulse has energy F_i (we use F to denote energy):

$$F_i = \int \frac{1}{2}(\epsilon_1|E|^2 + \mu_1|H|^2)dz = \epsilon_1 \int |A_i(z - c_1t)|^2 dz, \tag{22}$$

resulting in the relation

$$\int \mathcal{H} dz = \left(1 - \frac{v}{c_1}\right)F_i. \tag{23}$$

As time evolves and the incident pulse gets scattered from the modulated region, it gives rise to a reflected pulse and a transmitted pulse. These two pulses are separated in space so we can evaluate them individually. For the reflected pulse, we have

$$E_r(z,t) = A_r(z + c_1t), \quad H_r(z,t) = -\tfrac{1}{\eta_1}A_r(z + c_1)t. \tag{24}$$

For the transmitted pulse, we have

$$E_t(z,t) = A_t(z - c_2t), \quad H_t(z,t) = \tfrac{1}{\eta_2}A_t(z - c_2)t. \tag{25}$$

The total Hamiltonian after scattering is found to be

$$\int \mathcal{H} dz = \left(1 + \frac{v}{c_1}\right)F_r + \left(1 - \frac{v}{c_2}\right)F_t. \tag{26}$$

Since the total Hamiltonian $\int \mathcal{H}\, dz$ is conserved under time evolution, we obtain the following relation:

$$\left(1 - \frac{v}{c_1}\right)F_i = \left(1 + \frac{v}{c_1}\right)F_r + \left(1 - \frac{v}{c_2}\right)F_t. \tag{27}$$

To see how the preceding equation is related to photon-number conservation, we need to consider the carrier frequencies of the three pulses. Suppose that the incident pulse is a narrow band pulse centered at frequency ω_i. Each spectral component is in the form of a plane wave: $\exp[i(kz - \omega t)]$. Performing a Galilean transformation as indicated in Eq. (2), we obtain $\exp[i(kz - \omega t)] = \exp\{i[kz' - (\omega - kv)t']\}$. Thus, in the Galilean frame, the frequency of the wave is $\omega' = \omega - kv$, where ω and k are the frequency and wave number in the laboratory frame. Since the modulation is purely spatial in the Galilean frame, in this frame the frequencies of the incident, reflected, and transmitted waves should be the same. This leads to the following relation:

$$\frac{\omega_r - \omega_i}{k_r - k_i} = \frac{\omega_t - \omega_i}{k_t - k_i} = v. \tag{28}$$

This is also called a phase continuity relation [19] and holds for both the subluminal and superluminal cases.

Figure 2 shows the preceding relation graphically in the $\omega - k$ plane. Frequencies of three waves interacting with the moving index region must lie on the dashed line, which has a slope equal to the velocity v of traveling-wave modulation. The fourth crossing with the dashed line on the left represents a wave incident from region 2 and is not relevant here. From Eq. (28), frequencies of the reflected and transmitted waves are found to be

$$\omega_r = \frac{1 - v/c_1}{1 + v/c_1}\omega_i, \quad \omega_t = \frac{1 - v/c_1}{1 - v/c_2}\omega_i. \tag{29}$$

Using these frequencies, Eq. (27) can be written as

$$\frac{F_i}{\omega_i} = \frac{F_r}{\omega_r} + \frac{F_t}{\omega_t}. \tag{30}$$

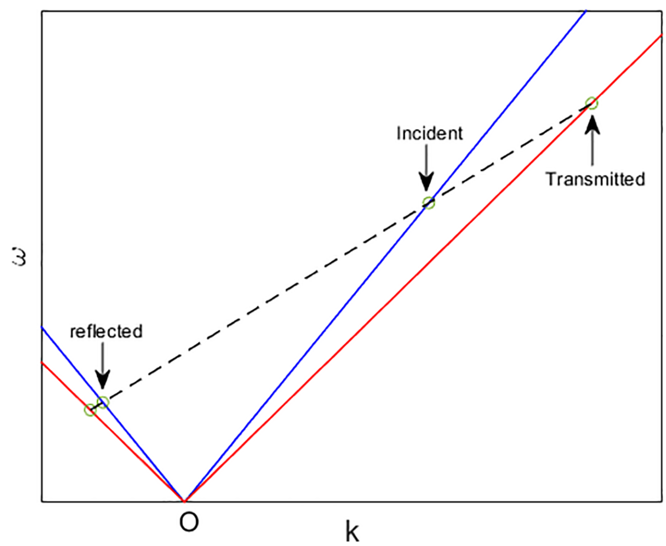

FIG. 2. Frequencies of the three waves (arrows) in the subluminal case indicated in the $\omega - k$ plane. Light lines in region 1 (blue) and region 2 (red) intersect with the dashed line, whose slope equals modulation speed.

Equation (30) can be related to the number of photons by dividing it by $\hbar$ on both sides. Recalling from quantum mechanics that $\hbar\omega_i$ is the energy of photons present in a wave of frequency ω_i, we obtain $N_i = N_r + N_t$ and find that the total number of photons in the incident light pulse is the same as the total number of photons in the reflected and transmitted pulses. Thus, we obtain the photon-number conservation law, showing that the number of photons is conserved even when energy is not. For this type of scattering, it makes sense to define the reflectivity and transmissivity as the ratio of photon numbers in the corresponding pulses:

$$R_{1\to 2} = \frac{F_r/\omega_r}{F_i/\omega_i} = \frac{1 + v/c_1}{1 - v/c_1}\frac{F_r}{F_i},$$
$$T_{1\to 2} = \frac{F_t/\omega_t}{F_i/\omega_i} = \frac{1 - v/c_2}{1 - v/c_1}\frac{F_t}{F_i}. \tag{31}$$

The subscript $1 \to 2$ indicates that the incident pulse goes from region 1 to region 2. It is easy to show that $R_{1\to 2} + T_{1\to 2} = 1$.

B. Superluminal case

The situation turns out to be different in the superluminal case. We can still follow the same procedure and evaluate the integral $\int \mathcal{H} dz$ before and after the modulated region. The difference from the subluminal case is that now both the reflected and transmitted pulses propagate in region 2 (see Fig. 1). Because of this difference, we obtain the following relation among energies of the incident, reflected, and transmitted pulses:

$$\left(\frac{v}{c_1} - 1\right)F_i = \left(\frac{v}{c_2} - 1\right)F_t - \left(\frac{v}{c_2} + 1\right)F_r. \tag{32}$$

To relate this expression to the number of photons in each pulse, we use the phase continuity relation in Eq. (28) and obtain the frequencies of transmitted and reflected waves in

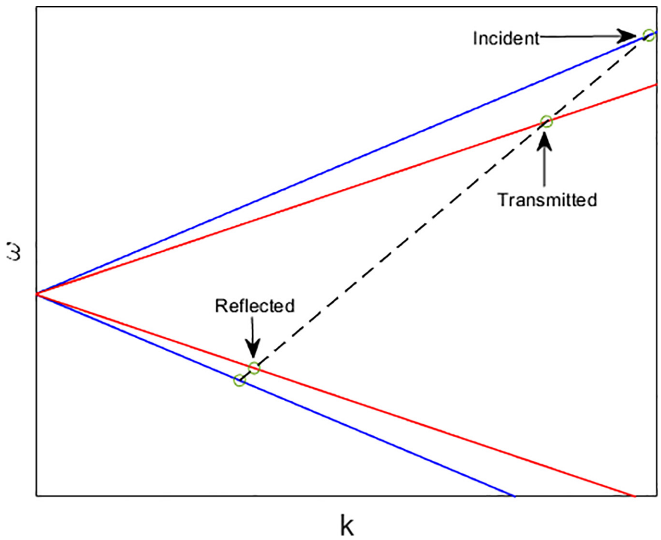

FIG. 3. Same as Fig. 2 but for the superluminal case.

the form

$$\omega_t = \frac{1/c_1 - 1/v}{1/c_2 - 1/v}\omega_i, \quad \omega_r = -\frac{1/c_1 - 1/v}{1/c_2 + 1/v}\omega_i. \tag{33}$$

To highlight the difference between subluminal and superluminal cases, we show the phase-continuity relation in Fig. 3, which should be compared with Fig. 2. Note that there is a fourth intersection point in Fig. 3 that is not labeled. This wave corresponds to an incident wave that travels in the backward direction, which is not present in the schematic shown in Fig. 1. We can immediately see that the reflected pulse actually has a negative frequency in the superluminal case. The presence of a negative frequency should not be surprising. Physical quantities are always real, and a wave in the form $\exp(ikz - i\omega t)$ is always accompanied by a wave of the form $\exp(-ikz + i\omega t)$. When we count the number of photons, we should use the absolute value of this frequency. As before, we can relate Eq. (32) to the number of photons as

$$\frac{F_i}{\omega_i} = \frac{F_t}{\omega_t} - \frac{F_r}{|\omega_r|}. \tag{34}$$

This equation shows that the total number of photons in the incident pulse is not conserved in the superluminal case. However, a different conservation law still exists. We can interpret Eq. (34) to show that the number of photons in the forward direction minus the number of photons in the backward direction is conserved before and after the scattering process. The physical interpretation for this is that temporal modulation can create photon pairs from vacuum energy but the two photons must travel in opposite directions. This is similar to the process of spontaneous frequency down-conversion in a nonlinear medium, which splits a single photon of energy $2\hbar\omega$ into two photons of energy $\hbar\omega$. We show later that the quantum descriptions of these two phenomena are actually the same.

Similar to the subluminal case, we can relate in the superluminal case the reflectivity and transmissivity to the ratio of photon numbers as

$$R_\pm = \frac{F_r/|\omega_r|}{F_i/\omega_i} = \frac{1/c_2 \pm 1/v}{1/c_1 \mp 1/v}\frac{F_r}{F_i},$$
$$T_\pm = \frac{F_t/\omega_t}{F_i/\omega_i} = \frac{1/c_2 \mp 1/v}{1/c_1 \mp 1/v}\frac{F_t}{F_i}. \tag{35}$$

However, it is their difference, rather than the sum, that is related as $T_\pm - R_\pm = 1$. Also, the subscripts $\pm$ correspond to the cases where the incident wave is traveling in the forward and backward directions.

It should be evident by now that the subluminal case and superluminal case are fundamentally different. In the first case, the number of photons is conserved during scattering, while in the second case, photon pairs can be generated, but they must propagate in opposite directions. One important consequence is that parametric amplification is only possible in the superluminal case, since the total energy of the wave in the subluminal case must be bounded. We can test the concept developed in this section by considering two limiting cases of purely spatial modulation ($v = 0$) and purely temporal modulation ($v \to \infty$).

In the spatial case, we set $v = 0$. It follows from Eq. (29) that frequencies ω_r and ω_t do not change from the incident frequency ω_i, and each photon's energy will not change during scattering. When the modulation is purely spatial, the scattering process should conserve energy. Since each photon's energy is conserved, the total photon number is also conserved. This is consistent with our results. In the case of purely temporal modulation ($v \to \infty$), the process conserves momentum, but not energy. Thus, photons can be generated from vacuum energy in pairs with the opposite momenta, i.e., they must travel in opposite directions, one forward and the other backward. Thus, it is the difference between the forward-moving photons and backward-moving photons that is conserved, as we found earlier in the superluminal case.

IV. RECIPROCITY

In the case of light scattering from a static structure, the principle of reciprocity holds and requires that the reflectivity and transmissivity of the structure remain the same whether light travels in the forward or backward direction. The question we ask is whether reciprocity holds in the case of scattering from a traveling-wave modulation.

A. Subluminal case

In this case, we can use Eq. (2) and work in a Galilean frame moving with the speed of modulation. In this frame, the modulation becomes purely spatial and we can use solutions of Eqs. (3) in the form of plane waves. When the incident plane wave travels from region 1 to region 2, the electric and magnetic fields in these two regions are given by

$$E_{1\to2}(z',t') = e^{-i\omega' t'}\begin{cases} e^{ik'_{+,1}z'} + \tilde{\rho} e^{ik'_{-,1}z'} & \text{for } z' \ll 0 \\ \tilde{\tau} e^{ik'_{+,2}z'} & \text{for } z' \gg 0, \end{cases} \tag{36}$$

$$H_{1\to2}(z',t') = e^{-i\omega' t'}\begin{cases} \frac{1}{\eta_1}[e^{ik'_{+,1}z'} - \tilde{\rho} e^{ik'_{-,1}z'}] & \text{for } z' \ll 0 \\ \frac{1}{\eta_2}\tilde{\tau} e^{ik'_{+,2}z'} & \text{for } z' \gg 0, \end{cases} \tag{37}$$

where the region of modulation begins at $z = 0$, and we only have a transmitted wave in region 2. All three waves share the same frequency ω' in the moving frame. The coefficients $\tilde{\rho}$ and $\tilde{\tau}$ relate the amplitudes of the reflected and transmitted fields to the input field and are functions of ω'.

The dispersion relation of a homogeneous medium is different in the moving frame from the laboratory frame. By performing a Galilean transform, it takes the form

$$k'_{\pm,m}(\omega') = \frac{\omega'}{\pm c_m - v}, \tag{38}$$

where $m = 1, 2$ represents different sides of the modulated region and the $\pm$ signs represent the forward and backward waves, respectively. We can construct narrow-band pulses from the monochromatic solution in Eqs. (36) and (37) if the spectrum of input pulse is narrow enough that the frequency dependence of $\tilde{\rho}$ and $\tilde{\tau}$ can be neglected.

The electric field of any incident pulse can be written as

$$E_i(z',t') = \int d\omega' \tilde{A}(\omega') e^{i(k'_{+,1}z' - \omega' t')}, \tag{39}$$

where $\tilde{A}(\omega')$ is the spectral amplitude of the pulse. Using the dispersion relation in Eq. (38), it can be written in the form $E_i(z',t') = A(t' - \frac{z'}{c_1 - v})$, where $A(t')$ is the inverse Fourier transform of $\tilde{A}(\omega')$. Transforming back into the original fixed frame, we obtain

$$E_i(z,t) = A\left(\frac{c_1 t - z}{c_1 - v}\right). \tag{40}$$

The electric fields of the reflected and transmitted pulses can be calculated in a similar fashion to obtain

$$E_r(z,t) = \tilde{\rho} A\left(\frac{z + c_1 t}{c_1 + v}\right), \quad E_t(z,t) = \tilde{\tau} A\left(\frac{c_2 t - z}{c_2 - v}\right). \tag{41}$$

By calculating the energy of pulses and using Eqs. (31), we obtain the following relations for the reflectivity and transmissivity:

$$R_{1\to2} = \frac{1 + v/c_1}{1 - v/c_1}\frac{F_r}{F_i} = \left(\frac{1 + v/c_1}{1 - v/c_1}\right)^2 |\tilde{\rho}|^2, \tag{42}$$

$$T_{1\to2} = \frac{1 - v/c_2}{1 - v/c_1}\frac{F_t}{F_i} = \frac{\eta_1}{\eta_2}\left(\frac{1 - v/c_2}{1 - v/c_1}\right)^2 |\tilde{\tau}|^2. \tag{43}$$

We use the same procedure for the case where incident wave is traveling from region 2 into region 1. First, we write the plane-wave solutions in the moving frame:

$$E_{2\to1} = e^{-i\omega' t'}\begin{cases} e^{ik'_{-,2}z'} + \tilde{\rho}' e^{ik'_{+,2}z'} & \text{for } z' \gg 0 \\ \tilde{\tau}' e^{ik'_{-,1}z'} & \text{for } z' \ll 0, \end{cases} \tag{44}$$

$$H_{2\to1} = e^{-i\omega' t'}\begin{cases} -\frac{1}{\eta_2}[e^{ik'_{-,2}z'} + \tilde{\rho}' e^{ik'_{+,2}z'} & \text{for } z' \gg 0 \\ -\frac{1}{\eta_1}\tilde{\tau}' e^{ik'_{-,1}z'}, & \text{for } z' \ll 0. \end{cases} \tag{45}$$

Then, we can relate the coefficients $\tilde{\rho}'$ and $\tilde{\tau}'$ to the reflectivity and transmissivity as

$$R_{2\to1} = \left(\frac{1 - v/c_2}{1 + v/c_2}\right)^2 |\tilde{\rho}'|^2, \tag{46}$$

$$T_{2\to1} = \frac{\eta_2}{\eta_1}\left(\frac{1 + v/c_1}{1 + v/c_2}\right)^2 |\tilde{\tau}'|^2. \tag{47}$$

In the general case when plane waves are incident from both sides of the modulated region, we can find the solution by considering superposition in the form

$$\begin{pmatrix} E(z',t') \\ H(z',t') \end{pmatrix} = a_+ \begin{pmatrix} E_{1\to2} \\ H_{1\to2} \end{pmatrix} + a_- \begin{pmatrix} E_{2\to1} \\ H_{2\to1} \end{pmatrix}, \tag{48}$$

where $a_\pm$ are two arbitrary complex numbers representing relative amplitudes of the incident fields. Since this linear superposition is a solution to the original Maxwell's equation, it must satisfy the conservation law that we derived earlier.

Consider the generalized Poynting's theorem in Eq. (16). For the preceding solution, $\mathcal{H}$ is independent of t', making $\partial\mathcal{H}/\partial t' = 0$. Thus, the quantity inside the bracket on the right side of Eq. (16) should be independent of z'. We can explicitly evaluate this quantity for $z' \ll 0$ and for $z' \gg 0$, and they must agree. Furthermore, they must agree for any value of $a_\pm$. This consideration leads to the following three relations:

$$\left(\frac{1 + v/c_1}{1 - v/c_1}\right)^2 |\tilde{\rho}|^2 + \frac{\eta_1}{\eta_2}\left(\frac{1 - v/c_2}{1 - v/c_1}\right)^2 |\tilde{\tau}|^2 = 1, \tag{49}$$

$$\left(\frac{1 - v/c_2}{1 + v/c_2}\right)^2 |\tilde{\rho}'|^2 + \frac{\eta_2}{\eta_1}\left(\frac{1 + v/c_1}{1 + v/c_2}\right)^2 |\tilde{\tau}'|^2 = 1, \tag{50}$$

$$\frac{(1 + v/c_1)^2}{\eta_1}\tilde{\rho}\tilde{\tau}'^* + \frac{(1 - v/c_2)^2}{\eta_2}\tilde{\rho}'^*\tilde{\tau} = 0. \tag{51}$$

The first two equations are equivalent to the photon-number conservation in the forward and backward directions. The last equation relates the phases of these coefficients. Using Eq. (51) and some algebra, we find the following relation:

$$R_{1\to2} = R_{2\to1}, \quad T_{1\to2} = T_{2\to1}. \tag{52}$$

This is the statement of reciprocity under subluminal modulation.

B. Superluminal case

Similar to the subluminal case, we can also find a relation between the reflectivities and transmissivities for the plane waves incident from opposite directions. We use the following transformation in the superluminal case:

$$\xi = z, \quad \tau = t - z/v. \tag{53}$$

In this frame, we redefine the origin of time at different locations based on when the modulation arrives at this location. We call this frame the superluminal frame, in which

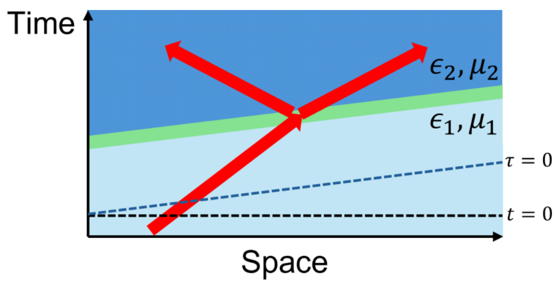

FIG. 4. The initial condition in the laboratory frame and the superluminal frame.

the space-time modulation becomes purely temporal. Note that the symbol τ represents the time in the superluminal frame, and it is distinct from the reflection coefficient, which is denoted as $\tilde{\tau}$.

We are interested in finding the interaction of an incident pulse with a superluminal modulation as shown in Fig. 4. In the laboratory frame, we will specify the input light field at $t = 0$, and the evolution of the field with respect to t can be solved using Maxwell's equations. In the superluminal frame, the initial condition needs to be specified for $\tau = 0$. From the consideration of causality, it is clear that the field at $\tau = 0$ is completely determined by the incident field at $t = 0$. With the initial condition in the superluminal frame known, the evolution of the field can be equivalently studied in this new frame.

Maxwell's equations in Eqs. (1) take the following form in the superluminal frame:

$$\frac{\partial}{\partial \tau}\left(\epsilon(\tau)E - \frac{H}{v}\right) = -\frac{\partial H}{\partial \xi},$$
$$\frac{\partial}{\partial \tau}\left(\mu(\tau)H - \frac{E}{v}\right) = -\frac{\partial E}{\partial \xi}. \quad (54)$$

By transforming Eq. (17) into the superluminal frame, we obtain

$$\frac{\partial}{\partial \tau}\left[\epsilon|E|^2 + \mu|H|^2 - \frac{1}{2}\left(v\epsilon\mu + \frac{1}{v}\right)(E^*H + EH^*)\right] = -\frac{\partial}{\partial \xi}\left[\frac{1}{2}(E^*H + EH^*) - \frac{v}{2}(\epsilon|E|^2 + \mu|H|^2)\right]. \quad (55)$$

Consider first the case of a plane wave traveling forward in region 1 before it is scattered by the modulated region. Since the modulation is purely temporal in the superluminal frame, the wave vector should be conserved. Using the symbols κ and Ω for the wave number and frequency of the waves in this frame, we can write the solution as

$$E_+ = e^{i\kappa\xi}\begin{cases} e^{-i\Omega_{1,+}\tau} \text{ for } \tau \ll 0 \\ \tilde{\tau} e^{-i\Omega_{2,+}\tau} + \tilde{\rho} e^{-i\Omega_{2,-}\tau} \text{ for } \tau \gg 0, \end{cases} \quad (56)$$

$$H_+ = e^{i\kappa\xi}\begin{cases} \frac{1}{\eta_1} e^{-i\Omega_{1,+}\tau} \text{ for } \tau \ll 0 \\ \frac{1}{\eta_2}[\tilde{\tau} e^{-i\Omega_{2,+}\tau} - \tilde{\rho} e^{-i\Omega_{2,-}\tau}] \text{ for } \tau \gg 0, \end{cases} \quad (57)$$

where $\tilde{\rho}$ and $\tilde{\tau}$ are the transmission and reflection coefficients and $\Omega_\pm$ are the frequencies (with respect to τ) in the forward and backward directions, respectively. Their relation to κ can be obtained by writing the dispersion relation in the superluminal frame:

$$\Omega_\pm = \frac{\kappa}{\pm 1/c - 1/v}, \quad (58)$$

where c is the speed of light (in the laboratory frame) in the medium in which the wave is propagating.

In the case of a plane wave incident from region 2, the solution takes the form

$$E_- = e^{i\kappa\xi}\begin{cases} e^{-i\Omega_{1,-}\tau}, \text{ for } \tau \ll 0 \\ \tilde{\tau}' e^{-i\Omega_{2,-}\tau} + \tilde{\rho}' e^{-i\Omega_{2,+}\tau}, \text{ for } \tau \gg 0, \end{cases} \quad (59)$$

$$H_- = e^{i\kappa\xi}\begin{cases} -\frac{1}{\eta_1} e^{-i\Omega_{1,-}\tau} \text{ for } \tau \ll 0 \\ \frac{1}{\eta_2}[-\tilde{\tau}' e^{-i\Omega_{2,-}\tau} + \tilde{\rho}' e^{-i\Omega_{2,+}\tau}] \text{ for } \tau \gg 0. \end{cases} \quad (60)$$

We now relate the reflection and transmission coefficients to the reflectivity and transmissivity defined in Eq. (35). As before, we construct a narrow-band pulse from the plane waves,

$$E(\xi, \tau) = \int \tilde{A}(\kappa)E_+(\xi, \tau; \kappa)d\kappa, \quad (61)$$

where $E_+(\xi, \tau; \kappa)$ is the forward incident field with the wave vector κ. Following the approach used in the subluminal case, we obtain

$$R_+ = \frac{\eta_1(1/c_2 + 1/v)^2}{\eta_2(1/c_1 - 1/v)^2}|\tilde{\rho}|^2, \quad (62)$$

$$T_+ = \frac{\eta_1(1/c_2 - 1/v)^2}{\eta_2(1/c_1 - 1/v)^2}|\tilde{\tau}|^2, \quad (63)$$

In the case of a backward propagating incident pulse, we obtain

$$R_- = \frac{\eta_1(1/c_2 - 1/v)^2}{\eta_2(1/c_1 + 1/v)^2}|\tilde{\rho}'|^2, \quad (64)$$

$$T_- = \frac{\eta_1(1/c_2 + 1/v)^2}{\eta_2(1/c_1 + 1/v)^2}|\tilde{\tau}'|^2. \quad (65)$$

To find a relation between R_- and R_+, we construct a superposition of incident waves from both directions,

$$E = a_+E_+ + a_-E_-, \quad (66)$$

and focus on Eq. (55). The right side vanishes because it does not depend on ξ. Thus, the left side must also be 0, implying that the term inside the bracket does not change with τ. We explicitly evaluate it for $\tau \ll 0$ and $\tau \gg 0$ and employ the equality requirement for arbitrary a_+ and a_-. As in the subluminal case, this leads to the following three relations:

$$\frac{\eta_1(1/c_2 - 1/v)^2}{\eta_2(1/c_1 - 1/v)^2}|\tilde{\tau}|^2 - \frac{\eta_1(1/c_2 + 1/v)^2}{\eta_2(1/c_1 - 1/v)^2}|\tilde{\rho}|^2 = 1, \quad (67)$$

$$\frac{\eta_1(1/c_2 + 1/v)^2}{\eta_2(1/c_1 + 1/v)^2}|\tilde{\tau}'|^2 - \frac{\eta_1(1/c_2 - 1/v)^2}{\eta_2(1/c_1 + 1/v)^2}|\tilde{\rho}'|^2 = 1, \quad (68)$$

$$(1/c_2 + 1/v)^2\tilde{\rho}^*\tilde{\tau}' = (1/c_2 - 1/v)^2\tilde{\tau}^*\tilde{\rho}'. \quad (69)$$

The first two relations are simply equivalent to $T_\pm - R_\pm = 1$. Using the third relation, we prove the reciprocity relations in the superluminal case:

$$R_+ = R_-, \quad T_+ = T_-. \tag{70}$$

To conclude, even though there is a preferred direction in a space-time modulated system, the conservation law that we have derived still implies reciprocity in both the subluminal and superluminal cases. It is worth recalling that the reflectivity and transmissivity are defined in terms of a ratio based on the number of photons in each pulse.

V. QUANTUM OPTICAL FORMULATION

In this section, we use quantum mechanics to formulate the scattering problem in the subluminal and superluminal cases. The photon-number conservation law appears naturally in this formulation.

A. Subluminal case

In the subluminal case, we implement quantization in the moving frame, and then convert it into the laboratory frame using Galilean transformation. This makes physical sense, because the system is time invariant in a Galilean frame, which allows us to use the normal modes [see Eq. (11)].

The first step that we take is to consider quantizing the field in a homogeneous medium (characterized by ϵ and μ) in the moving frame, and compare it to the result of quantizing in the laboratory frame (which is a standard procedure) to explore the transformation of the field operators between the two frames, and more importantly, to show that the quantization procedure that we take in the moving frame is indeed consistent with the standard approach of quantization of the field in the laboratory frame. Essentially, we would need the total number of photons after the two quantization procedures to be the same.

In the moving frame, the conserved quantity is the effective Hamiltonian whose density is given in Eq. (15). We write the effective Hamiltonian as

$$H_{\text{eff}} = \frac{A_e}{2}\int \epsilon|E|^2 + \mu|H|^2 - v\epsilon\mu(E^*H + EH^*)dz'. \tag{71}$$

Since the Hamiltonian should have a unit of energy, we have introduced an effective area A_e for the cross section in this one-dimensional problem. In a homogeneous medium, normal modes are plane waves. Using them, we can write the electric and magnetic fields in the form

$$E(z',t') = \sum_{s=\pm}\int_0^\infty d\omega' E_s(\omega')[a'_s(\omega')e^{i(k'_s z'-\omega' t')} + a'^*_s(\omega')e^{-i(k'_s z'-\omega' t')}], \tag{72}$$

$$H(z',t') = \sum_{s=\pm}\frac{s}{\eta}\int_0^\infty d\omega' E_s(\omega')(a'_s(\omega')e^{i(k'_s z'-\omega' t')} + a'^*_s(\omega')e^{-i(k'_s z'-\omega' t')}), \tag{73}$$

where $s = \pm$ represents the forward and backward propagating modes. When we write s as if it is a number, we use ± 1 and for $s = \pm$. Note that $E_s(\omega')$ are real functions.

Using the preceding two modal expansions, we evaluate the effective Hamiltonian given in Eq. (71). After lengthy algebra, we obtain

$$H_{\text{eff}} = \sum_{s=\pm}\int_0^\infty d\omega' 2\pi\epsilon c A_e(1 - sv/c)^2 E_s^2(\omega')(a'_s a'^*_s + a'^*_s a'_s). \tag{74}$$

We choose the normalization constants $E_s(\omega')$ in the standard way to perform field quantization [20]:

$$E_s(\omega') = \left(\frac{\hbar\eta\omega'}{4\pi A_e}\right)^{1/2}\frac{1}{1 - sv/c}. \tag{75}$$

Then, the effective Hamiltonian takes the standard form

$$H_{\text{eff}} = \sum_s \int_0^\infty d\omega' \frac{\hbar\omega'}{2}(a'_s a'^*_s + a'^*_s a'_s), \tag{76}$$

and $|a'_s(\omega')|^2$ can be viewed as the photon number density at one frequency. At this point, we promote $a'_s(\omega')$ to an annihilation operator with the commutation relation

$$[\hat{a}'_s(\omega'), \hat{a}'^\dagger_{s'}(\omega'')] = \delta_{ss'}\delta(\omega' - \omega''), \tag{77}$$

which leads to the correct form of Heisenberg's equation of motion. Next, we investigate how the annihilation operators transform between the original and the moving frames. In the laboratory frame, $v = 0$, and the electric field operator should be given by

$$\hat{E}(z,t) = \sum_{s=\pm}\int_0^\infty d\omega\left(\frac{\hbar\eta\omega}{4\pi A_e}\right)^{\frac{1}{2}}[\hat{a}_s(\omega)e^{i[k_s(\omega)z-\omega t]} + \hat{a}^\dagger_s(\omega)e^{-i[k_s(\omega)z-\omega t]}], \tag{78}$$

where $k_s(\omega) = s\omega/c$ and the commutation relation is given by

$$[\hat{a}_s(\omega_1), \hat{a}^\dagger_{s'}(\omega_2)] = \delta_{ss'}\delta(\omega_1 - \omega_2). \tag{79}$$

We can also perform a Galilean transform of Eq. (72) by substituting $z' = z - vt$, $t' = t$. Then, by comparing the result with Eq. (78), we obtain the following relation:

$$\hat{a}_s(\omega) = \left(1 - \frac{sv}{c}\right)^{1/2}\hat{a}'_s[(1 - sv/c)\omega]. \tag{80}$$

It is easy to verify that this transformation does lead to the correct commutation relation. Thus, the number of photons in the original and moving frames remains the same. Therefore, the quantization procedure in the moving frame is fully consistent with the standard quantization in the laboratory frame.

Next, we perform the quantization of the scattering problem. In our scattering problem, a space-time modulated region separates two homogeneous media. In this case, the normal modes are not plane waves anymore, but instead they are the plane-wave solutions of the scattering problem given in Eqs. (36) and (37) and in Eqs. (44) and (45). These modes contain an incident wave, a reflected wave, and a transmitted wave. We can replace the plane waves in Eq. (72) with these normal modes, and this gives us the quantized field, which is written as

$$\hat{E}(z',t') = \int_0^\infty d\omega' \left(\frac{\hbar\omega'\eta_1}{4\pi A_e}\right)^{\frac{1}{2}} \frac{\hat{a}'_+(\omega')e^{-i\omega't'}}{1-v/c_1} \times \begin{cases} e^{ik'_{+,1}z'} + \tilde{\rho}e^{ik'_{-,1}z'}, & \text{for } z' \ll 0 \\ \tilde{\tau}e^{ik'_{+,2}z'}, & \text{for } z' \gg 0 \end{cases} + \int_0^\infty d\omega' \left(\frac{\hbar\omega'\eta_2}{4\pi A_e}\right)^{\frac{1}{2}} \frac{\hat{a}'_-(\omega')e^{-i\omega't'}}{1+v/c_2} \times \begin{cases} \tilde{\tau}'e^{ik'_{-,1}z'}, & \text{for } z' \ll 0 \\ e^{ik'_{-,2}z'} + \tilde{\rho}'e^{ik'_{+,2}z'}, & \text{for } z' \gg 0 \end{cases} + \text{H.c.}, \quad (81)$$

where $\pm$ denote the forward and backward directions. Here, the operators correspond to the incident waves. "H.c." represents Hermitian conjugate. The operator $\hat{a}'_+$ corresponds to the incident wave, moving forward in region 1. Thus, we should use the parameters in region 1 for the first integral. The operator $\hat{a}'_-$ corresponds to an incident wave moving backward in region 2, and we use parameters of region 2 in the second part of the electric field.

Scattering of the incident wave by the modulate region produces two outgoing waves. We denote the creation operators for these waves by $\hat{b}'_+(\omega')$ and $\hat{b}'_-(\omega')$. More specifically, $\hat{b}'_+(\omega')$ is for the output wave traveling forward in region 2 and $\hat{b}'_-(\omega')$ is for the output wave traveling backward in region 1. Thus, the electric field operator in these two regions is given by

$$\hat{E}_{\text{out}}(z' \ll 0, t') = \int_0^\infty d\omega' \left(\frac{\hbar\omega'\eta_1}{4\pi A_e}\right)^{\frac{1}{2}} \frac{\hat{b}'_-(\omega')e^{i(k'_{-,1}z'-\omega't')}}{1+v/c_1} + \text{H.c.}, \quad (82)$$

$$\hat{E}_{\text{out}}(z' \gg 0, t') = \int_0^\infty d\omega' \left(\frac{\hbar\omega'\eta_2}{4\pi A_e}\right)^{\frac{1}{2}} \frac{\hat{b}'_-(\omega')e^{i(k'_{+,2}z'-\omega't')}}{1-v/c_2} + \text{H.c.} \quad (83)$$

By comparing these expressions with the outgoing components in Eq. (81), we obtain the following relations between the operators of incoming and outgoing waves:

$$\hat{b}'_+(\omega') = \left(\frac{\eta_1}{\eta_2}\right)^{\frac{1}{2}} \frac{1-v/c_2}{1-v/c_1}\tilde{\tau}\hat{a}'_+(\omega') + \frac{1-v/c_2}{1+v/c_2}\tilde{\rho}'\hat{a}'_-(\omega'),$$
$$\hat{b}'_-(\omega') = \left(\frac{\eta_2}{\eta_1}\right)^{\frac{1}{2}} \frac{1+v/c_1}{1+v/c_2}\tilde{\tau}'\hat{a}'_-(\omega') + \frac{1+v/c_1}{1-v/c_1}\tilde{\rho}\hat{a}'_+(\omega'). \quad (84)$$

The preceding two relations provide us with the transformation laws relating the incident and output waves of our scattering problem. Using the properties derived in Eqs. (49)–(51), we can show that the transformations in Eq. (84) are indeed unitary and preserve the correct commutation relations. In the Galilean frame, the transformation from incident-field operators to outgoing-field operators is identical to the quantum description of a beam splitter. It follows that the total number of photons will be conserved. The transformation back into the laboratory frame can be done with the help of Eq. (80). As seen in the laboratory frame, the mode of each photon will change due to the frequency shift caused by space-time modulation. As indicated by Eq. (29), the frequency bandwidth of the reflected and transmitted pulses will be different from that of the incident pulse. This implies that the pulse duration of the photon wave packet will also change after scattering from the moving modulation.

B. Superluminal case

In this case, we need to work in a superluminal frame [Eq. (53)]. As modulation becomes purely temporal in this frame, one cannot introduce global normal modes. The approach we take is to perform quantization in the superluminal frame before and after the modulation, where the medium is time independent. Transformation of the field before the modulation to the field after modulation is achieved through the reflection and transmission coefficients.

Similar to the approaches taken in the subliminal case, we first need to consider quantization of the field in the superluminal frame for a homogeneous time-independent medium and show that it is consistent with the quantization done in the laboratory frame. For this purpose, we first convert Maxwell's equation given in Eq. (1) to the superluminal frame and write them in a matrix form as

$$i\frac{\partial}{\partial\tau}\begin{pmatrix} \epsilon & -1/v \\ -1/v & \mu \end{pmatrix}\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} 0 & \hat{p}_\xi \\ \hat{p}_\xi & 0 \end{pmatrix}\begin{pmatrix} E \\ H \end{pmatrix}. \quad (85)$$

Using a technique similar to that used in Sec. II, the Hamiltonian of the system, for static values of ϵ and μ, can be written as

$$H_{\text{eff}} = \frac{A_e}{2}\int \left[\epsilon|E|^2 + \mu|H|^2 - \frac{1}{v}(E^*H + EH^*)\right]d\xi. \quad (86)$$

Adopting a procedure used in the subliminal case, we obtain the electric field operator in the form

$$\hat{E}(\xi,\tau) = \sum_{s=\pm}\int_0^\infty d\Omega \sqrt{\frac{\hbar\eta\Omega}{4\pi A_e}}\hat{a}_s(\Omega)e^{i[\kappa_s(\Omega)\xi-\Omega\tau]} + \text{H.c.}, \quad (87)$$

where the wave vector is given by

$$\kappa_s = \left(\frac{s}{c} - \frac{1}{v}\right)\Omega. \quad (88)$$

A similar expression can be written for the magnetic field. Using these results, the Hamiltonian becomes

$$\hat{H}_{\text{eff}} = \sum_{s=\pm}\int_0^\infty d\Omega \frac{\hbar\Omega}{2}[\hat{a}_s^\dagger(\Omega)\hat{a}_s(\Omega) + \hat{a}_s(\Omega)\hat{a}_s^\dagger(\Omega)]. \quad (89)$$

As before, the annihilation and creation operators satisfy the commutation relation

$$[\hat{a}_s(\Omega), \hat{a}_{s'}^\dagger(\Omega')] = \delta_{ss'}\delta(\Omega - \Omega'). \quad (90)$$

Transforming back Eq. (87) to the laboratory frame, we find that $a_s(\Omega)$ is also the operator in that frame. This is because the wave's frequency remains the same in the two frames. The photon number also remains unchanged. Thus, the quantization procedure in a superluminal frame is fully consistent with the standard quantization process.

Since the wave number κ is conserved in a superluminal frame, we should introduce annihilation operators as a function of κ. This can be done by simply performing a change of variables, and the result is

$$\hat{\alpha}(\kappa) = \begin{cases} \dfrac{1}{\sqrt{1/c - 1/v}}\hat{a}_+\left(\dfrac{\kappa}{1/c - 1/v}\right), & \kappa > 0, \\ \dfrac{1}{\sqrt{1/c + 1/v}}\hat{a}_-\left(\dfrac{\kappa}{-1/c - 1/v}\right), & \kappa < 0, \end{cases} \tag{91}$$

where positive and negative κ stand for the forward- and backward-traveling waves. The frequency Ω is related to κ by the following relations:

$$\Omega(\kappa) = \begin{cases} \Omega_+(\kappa) = \dfrac{\kappa}{1/c - 1/v}, & \kappa > 0, \\ \Omega_-(\kappa) = -\dfrac{\kappa}{1/c + 1/v}, & \kappa < 0. \end{cases} \tag{92}$$

Note that $\Omega(\kappa)$ is always positive as it is related to the photon's energy. The κ-dependent operators satisfy the following commutation relation:

$$[\hat{\alpha}(\kappa), \hat{\alpha}^\dagger(\kappa')] = \delta(\kappa - \kappa'). \tag{93}$$

In terms of the κ-dependent operators, the electric field operator becomes

$$\hat{E}(\xi, \tau) = \int_{-\infty}^{0} d\kappa \sqrt{\frac{\hbar\eta|\kappa|}{4\pi A_e}} \frac{\hat{\alpha}(\kappa)}{1/c + 1/v} e^{i[\kappa\xi - \Omega(\kappa)\tau]} + \int_{0}^{\infty} d\kappa \sqrt{\frac{\hbar\eta|\kappa|}{4\pi A_e}} \frac{\hat{\alpha}(\kappa)}{1/c - 1/v} e^{i[\kappa\xi - \Omega(\kappa)\tau]} + \text{H.c.} \tag{94}$$

Next, we bring back the impact of modulation. Since the postmodulation field can be calculated in terms of the premodulation field through the reflection and transmission coefficients as done in Sec. IV B, we expect that postmodulation field operators can be calculated in terms of premodulation field operators in the same way. Let $\hat{\alpha}(\kappa)$ be the premodulation annihilation operator. We can write the electric field before modulation as

$$\hat{E}(\xi, \tau \ll 0) = \sqrt{\frac{\hbar\eta_1|\kappa|}{4\pi A_e}} \left[\int_{-\infty}^{0} d\kappa \left(\frac{\hat{\alpha}(\kappa) e^{-i\Omega_{1,-}(\kappa)\tau}}{1/c_1 + 1/v} + \frac{\hat{\alpha}^\dagger(-\kappa) e^{i\Omega_{1,+}(-\kappa)\tau}}{1/c_1 - 1/v} \right) e^{i\kappa\xi} + \int_{0}^{\infty} d\kappa \left(\frac{\hat{\alpha}(\kappa) e^{-i\Omega_{1,+}(\kappa)\tau}}{1/c_1 - 1/v} + \frac{\hat{\alpha}^\dagger(-\kappa) e^{i\Omega_{1,-}(-\kappa)\tau}}{1/c_1 + 1/v} \right) e^{i\kappa\xi} \right]. \tag{95}$$

The frequencies $\Omega_\pm(\kappa)$ are given in Eq. (58). We also have $\Omega_\pm(-\kappa) = -\Omega_\pm(\kappa)$. The wave vector is conserved in such an interaction, As a result, the component with spatial dependence of $e^{i\kappa\xi}$ has contributions from both the annihilation operator $\hat{\alpha}(\kappa)$ and the creation operator $\hat{\alpha}^\dagger(-\kappa)$. Equations (56) and (59) show us how the input field evolves after modulation. Using them, we can write the electric field as

$$\hat{E}(\xi, \tau \gg 0) = \sqrt{\frac{\hbar\eta_1|\kappa|}{4\pi A_e}} \left\{ \int_{-\infty}^{0} d\kappa \left[\frac{\hat{\alpha}(\kappa)}{1/c_1 + 1/v} (\tilde{\tau}' e^{-i\Omega_{2,-}\tau} + \tilde{\rho}' e^{-i\Omega_{2,+}\tau}) + \frac{\hat{\alpha}^\dagger(-\kappa)}{1/c_1 - 1/v} (\tilde{\tau} e^{-i\Omega_{2,+}\tau} + \tilde{\rho} e^{-i\Omega_{2,-}\tau}) \right] e^{i\kappa\xi} + \int_{0}^{\infty} d\kappa \left[\frac{\hat{\alpha}(\kappa)}{1/c_1 - 1/v} (\tilde{\tau} e^{-i\Omega_{2,+}\tau} + \tilde{\rho} e^{-i\Omega_{2,-}\tau}) + \frac{\hat{\alpha}^\dagger(-\kappa)}{1/c_1 + 1/v} (\tilde{\tau}' e^{-i\Omega_{2,-}\tau} + \tilde{\rho}' e^{-i\Omega_{2,+}\tau}) \right] e^{i\kappa\xi} \right\}. \tag{96}$$

To clarify, all reflection and transmission coefficients ($\tilde{\rho}$, $\tilde{\tau}$, etc.) depend on κ and the argument should all be κ. If we define the postmodulation annihilation operator as $\hat{\beta}(\kappa)$, it determines the electric field after modulation. We expect the field for $\tau \gg 0$ to have the same form as in Eq. (96), after we replace $\hat{\alpha}(\kappa)$ with $\hat{\beta}(\kappa)$, and use ϵ_2 and μ_2 for the medium parameters. Comparing this equation with Eq. (98), we obtain the following relations between the annihilation operators before and after modulation:

$$\hat{\beta}(-\kappa) = \sqrt{\frac{\eta_1}{\eta_2}} \frac{1/c_2 + 1/v}{1/c_1 + 1/v} \tilde{\tau}'(-\kappa)\hat{\alpha}(-\kappa) + \sqrt{\frac{\eta_1}{\eta_2}} \frac{1/c_2 + 1/v}{1/c_1 - 1/v} \tilde{\rho}(-\kappa)\hat{\alpha}^\dagger(\kappa), \tag{97}$$

$$\hat{\beta}(\kappa) = \sqrt{\frac{\eta_1}{\eta_2}} \frac{1/c_2 - 1/v}{1/c_1 - 1/v} \tilde{\tau}(\kappa)\hat{\alpha}(\kappa) + \sqrt{\frac{\eta_1}{\eta_2}} \frac{1/c_2 - 1/v}{1/c_1 + 1/v} \tilde{\rho}'(\kappa)\hat{\alpha}^\dagger(-\kappa). \tag{98}$$

In the expression above, the variable κ is positive, such that $\hat{\beta}(-\kappa)$ stands for the operator of a backward moving wave. By taking the complex conjugate of Eqs. (56) and (59), we obtain the following relations:

$$\tilde{\rho}(-\kappa) = \tilde{\rho}^*(\kappa), \quad \tilde{\tau}(-\kappa) = \tilde{\tau}^*(\kappa). \tag{99}$$

The same relations hold for $\tilde{\rho}'$ and $\tilde{\tau}'$. Then, using the reciprocity relations derived in the last section, we can show that transformation from $\hat{\alpha}(\pm\kappa)$ to the operators $\hat{\beta}(\pm\kappa)$ is unitary. Furthermore, it has the same structure as found for the nonlinear process of degenerate frequency down-conversion [20]. Thus, photons are always generated in pairs (one in each mode), and the photon number difference between the two modes is conserved during the interaction. Indeed, one can show the following relation holds:

$$\hat{\beta}^\dagger(\kappa)\hat{\beta}(\kappa) - \hat{\beta}^\dagger(-\kappa)\hat{\beta}(-\kappa) = \hat{\alpha}^\dagger(\kappa)\hat{\alpha}(\kappa) - \hat{\alpha}^\dagger(-\kappa)\hat{\alpha}(-\kappa). \tag{100}$$

This is the quantum version of photon-number conservation found classically in Eq. (34). The quantum formulations permit us to consider the spontaneous version of this problem for which the input state is the vacuum state. In this case, photon pairs are generated in two modes propagating in the forward and backward directions. After the modulation, the average number of photons in each mode is $R_+ = R_-$, where R_+ is given in Eq. (62).

VI. CONCLUSIONS

In this work, we have discussed the propagation of electromagnetic waves in a space-time modulated medium, for which neither energy nor momentum remains conserved, because both the space and translation symmetries are broken. By writing Maxwell's equation in the form of a Schrödinger equation, we have found a conservation law in the specific case where modulation moves inside the medium at a constant speed v in a traveling-wave fashion. We find that the conserved quantity is a linear combination of the energy and momentum of the wave. We use this law to study how light is scattered inside such a medium. We consider both the subluminal ($v < c$) and superluminal ($v > c$) cases and find that they differ considerably in terms of the quantity that remains conserved. We show that the total number of photons remains conserved in the subluminal case. In contrast, it is the difference between the number of forward- and backward-moving photons that remains conserved in the superluminal case. We also study the reciprocity issue for a space-time modulated medium and find that reciprocity holds in all situations. We develop a quantum formulation of our scattering problem and show that the subluminal case is similar to the action of a beam splitter. In the superluminal case, photon pairs can be generated from vacuum in a fashion analogous to the two-mode frequency down-conversion process. Our analysis provides insight at a fundamental level into how photons interact with a medium whose permittivity and permeability vary in both space and time in a traveling-wave fashion. The results should be useful for experiments devoted to the propagation of electromagnetic beams in a space-time modulated medium.

Note added. Recently, we became aware of a preprint [21] where Liberal *et al.* found the same conservation law using Noether's theorem with a Lagrangian formulation. Our approach differs considerably from this work and adds more to this emerging field. We have addressed the implications of the discovered conservation law in both the subluminal and superluminal cases. Moreover, we have developed a quantum formulation that should prove quite valuable to further studies.

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