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Focusing of partially coherent light by a graded-index lens

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We use coherence theory to study how the focusing of an optical beam by a graded-index (GRIN) lens is affected when the incoming beam is only partially coherent. The Gaussian–Schell model is used to show that the intensity of a partially coherent beam exhibits self-imaging and evolves in a periodic fashion in a GRIN medium with a parabolic index profile. Spatial coherence of the beam affects a single parameter that governs how much the beam is compressed at the focal point. Our results show that the focal spot size depends on the fraction of the beam's diameter over which coherence persists. Focusing ceases to occur, and the beam may even expand at the focal point of a GRIN lens, when this fraction is below 10%.

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It is well known that a fully coherent optical beam can be focused by a lens down to a spot size comparable to the wavelength of incident light [1,2]. In practice, any incoming beam is likely to exhibit amplitude and phase fluctuations that render it partially coherent. Intuitively, one expects partial coherence to affect the spot size at the focal point. For this reason, considerable attention has been paid to the focusing of partially coherent beams by a lens [3–8].

A thin rod of a graded-index (GRIN) material also acts as a lens when its length is chosen suitably. Such devices, called GRIN lenses, have been available commercially since the 1970s and have found many applications because of their compact size with a flat surface. An optical beam, launched into a GRIN rod that is designed with a parabolic refractive-index profile, exhibits self-imaging such that it recovers all of its input properties periodically [9–12]. During each self-imaging period, a coherent beam undergoes a focusing phase such that its width is reduced considerably at the focal point where the width takes its minimum value. For a typical GRIN rod of 2 mm in diameter, the focal length is less than 1 cm, resulting in a compact flat lens.

Propagation of partially coherent beams inside a GRIN medium has been studied in several different contexts [13–19]. In a 1990 study, spectral modifications of a partially coherent beam were considered inside a GRIN medium [13]. In several later works, the focus was on changes in the polarization properties of a partially coherent beam propagating through a GRIN fiber [14–16]. In a 2015 study, attention was on the self-imaging phenomenon [17]. More recently, the evolution of a partially coherent vortex beam inside a GRIN medium has been discussed [18].

In this work, we consider how the focusing of an optical beam by a GRIN lens is affected when the incoming beam is only partially coherent. Similar to the case of a conventional lens, we expect the focal spot size to depend on the degree of coherence of the incoming beam. We first introduce the concept of cross-spectral density and provide mathematical details related to its evolution inside a GRIN medium. The Gaussian–Schell model is then used to study how the intensity of a partially coherent beam evolves inside the GRIN medium and to show that it changes in a periodic fashion that is analogous to a coherent beam. Spatial coherence of the beam affects a single parameter that governs how much the beam is compressed at the focal point. This feature is employed to study the dependence of the focal spot size on spatial coherence of the input beam. The main results are summarized at the end of the paper.

Consider a GRIN medium whose refractive index is designed to decrease radially in a parabolic fashion and can be written as

$$n(\rho) = n_0\left(1 - \frac{1}{2}b^2\rho^2\right), \quad (1)$$

where $\rho = \sqrt{x^2 + y^2}$ is the radial distance from the central axis of the GRIN rod (aligned with the z axis) and n_0 is the refractive index at $\rho = 0$. The index gradient b is defined as $b = \sqrt{2\Delta}/a$, where a is the radius of the GRIN rod's core and Δ is the relative core–cladding index difference.

It is easier to solve Maxwell's equations in the frequency domain by introducing the Fourier transform of the electric field $\mathbf{E}(\mathbf{r}, t)$ as

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, t)\, e^{i\omega t}\, dt. \quad (2)$$

Each component of $\tilde{\mathbf{E}}(\mathbf{r}, \omega)$ at a given frequency ω evolves inside the GRIN rod as [9–12]

$$\tilde{E}_j(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} K(\mathbf{r}, \mathbf{s})\tilde{E}_j(\mathbf{s}, \omega)\, d\mathbf{s}, \quad (3)$$

where $j = x, y,$ or z, $\mathbf{s}$ represents a point at the input plane $z = 0$, and the integral is over the surface of this plane. The propagation kernel can be written in terms of the elements of the ABCD matrix as

$$K(\mathbf{r},\mathbf{s}) = \left(\frac{ke^{ikz}}{2\pi iB}\right) \times \exp\left(\frac{ik}{2B}[D\,\mathbf{u}\cdot\mathbf{u} - 2\mathbf{u}\cdot\mathbf{s} + A\mathbf{s}\cdot\mathbf{s}]\right), \tag{4}$$

where $k = n_0(\omega/c)$ and $\mathbf{u} = (x, y)$ is the transverse part of the vector $\mathbf{r}$ at the location z. For a GRIN medium with the parabolic index profile in Eq. (1), the matrix elements are $A = D = \cos(bz)$ and $B = \sin(bz)/b$ [9,12]. For a homogeneous medium ($b = 0$), they become $A = D = 1$ and $B = z$. The element C is not needed but can be found from the condition $AD - BC = 1$, set by the requirement that the ABCD matrix must be unitary in the absence of losses.

The cross-spectral density is used for describing partially coherent light in the spectral domain. It represents a correlation function defined as [20]

$$W_{ij}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle \tilde{E}_i^*(\mathbf{r}_1,\omega)\tilde{E}_j(\mathbf{r}_2,\omega)\rangle, \tag{5}$$

where the average is over an ensemble in the frequency domain. The subscripts i, j can be dropped when the electric field is oriented along a specific direction in the transverse plane that does not change with propagation. We assume this to be the case and work with the scalar form of Eq. (5):

$$W(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle \tilde{E}^*(\mathbf{r}_1,\omega)\tilde{E}(\mathbf{r}_2,\omega)\rangle. \tag{6}$$

Using Eq. (3) in Eq. (6), the cross-spectral density at a distance z is related to its known form at $z = 0$ as

$$W(\mathbf{r}_1,\mathbf{r}_2,\omega) = \iint_{-\infty}^{\infty} K^*(\mathbf{r}_1,\mathbf{s}_1)K(\mathbf{r}_2,\mathbf{s}_2) W(\mathbf{s}_1,\mathbf{s}_2,\omega)\,d\mathbf{s}_1\,d\mathbf{s}_2. \tag{7}$$

Its evaluation requires a four-dimensional integration over the input-plane variables. Even though the algebra can be tedious, Eq. (7) provides a straightforward approach to study the evolution of partially coherent light inside a GRIN medium.

Before using Eq. (7), we need to specify the input cross-spectral density $W(\mathbf{s}_1,\mathbf{s}_2,\omega)$ at $z = 0$. The spectral degree of coherence between two points in the input plane is defined as [20]

$$\mu(\mathbf{s}_2-\mathbf{s}_1,\omega) = \frac{W(\mathbf{s}_1,\mathbf{s}_2,\omega)}{[S(\mathbf{s}_1,\omega)S(\mathbf{s}_2,\omega)]^{1/2}}, \tag{8}$$

where $S(\mathbf{s},\omega) = W(\mathbf{s},\mathbf{s},\omega)$ is the spectral intensity at the point $\mathbf{s}$. If the spectrum is the same at all spatial points, we can write spectral intensity in the form $S(\mathbf{s},\omega) = S_0(\omega)U(\mathbf{s})$, where $S_0(\omega)$ is the input spectrum and $U(\mathbf{s})$ is the dimensionless spatial profile of the beam's intensity. Combining Eqs. (7) and (8), we obtain

$$W(\mathbf{r}_1,\mathbf{r}_2,\omega) = S_0(\omega)\iint_{-\infty}^{\infty} K^*(\mathbf{r}_1,\mathbf{s}_1)K(\mathbf{r}_2,\mathbf{s}_2) \sqrt{U(\mathbf{s}_1)U(\mathbf{s}_2)}\mu(\mathbf{s}_2-\mathbf{s}_1,\omega)\,d\mathbf{s}_1\,d\mathbf{s}_2. \tag{9}$$

The spectral intensity of the beam at a distance z is obtained by setting $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}$ in Eq. (9).

Before we can use Eq. (9) to calculate propagation-induced changes in the intensity of an optical beam, we need to specify its spatial profile $U(\mathbf{s})$ at $z = 0$ as well as its degree of coherence $\mu(\mathbf{s}_2 - \mathbf{s}_1, \omega)$. In a well-known model, known as the Gaussian–Schell model [20], both are assumed to have a Gaussian shape:

$$U(\mathbf{s}) = \exp(-|\mathbf{s}|^2/w_0^2), \qquad \mu(\mathbf{s},\omega) = \exp(-|\mathbf{s}|^2/2\sigma_c^2), \tag{10}$$

where w_0 is the spot size of the input beam and σ_c is a measure of the distance over which spatial coherence of the beam persists.

The Gaussian–Schell model in Eq. (10) applies to a Gaussian beam whose spatial coherence decreases with distance in a Gaussian fashion. The beam's intensity at any point inside a GRIN medium is found from Eq. (9) by setting $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}$ in this equation:

$$S(\mathbf{r},\omega) = S_0(\omega)\iint_{-\infty}^{\infty} K^*(\mathbf{r},\ \mathbf{s}_1)K(\mathbf{r},\mathbf{s}_2) \sqrt{U(\mathbf{s}_1)U(\mathbf{s}_2)}\mu(\mathbf{s}_2-\mathbf{s}_1,\omega)\,d\mathbf{s}_1\,d\mathbf{s}_2. \tag{11}$$

Using the form of the kernel given in Eq. (4) and employing the Cartesian coordinates with $\mathbf{r} = (x, y, z)$ and $\mathbf{s}_j = (x'_j, y'_j)$ for $j = 1, 2$, Eq. (11) can be written as

$$S(\mathbf{r},\omega) = S_0(\omega)H(x,z)H(y,z), \tag{12}$$

where the function $H(x, z)$ is defined as

$$H(x,z) = \iint_{-\infty}^{\infty} \exp\left[-\frac{(x_1'^2) + x_2'^2)}{2w_0^2} - \frac{(x_1' - x_2')^2}{2\sigma_c^2}\right] K^*(x,x_1')K(x,x_2')\,dx_1'dx_2', \tag{13}$$

and the one-dimensional kernel $K(x, x')$ is obtained from Eq. (4):

$$K(x,x') = \sqrt{\frac{ke^{ikz}}{2\pi iB}}\exp\left[\frac{ik}{2B}(Dx^2 - 2xx' + Ax'^2)\right]. \tag{14}$$

It is possible to carry out the integrations in Eq. (13) in an analytic form. After considerable algebra, the beam's intensity at a distance z inside the GRIN rod can be written as [12,14]

$$I(\mathbf{r},\omega) = \frac{S(\mathbf{r},\omega)}{S_0(\omega)} = \frac{w_0}{w_e(z)}\exp\left[-\frac{(x^2+y^2)}{w_e^2(z)}\right], \tag{15}$$

where the effective width of the beam at a distance z is given as

$$w_e(z) = w_0\sqrt{f(z)}, \quad f(z) = \cos^2(bz) + C_p^2\sin^2(bz). \tag{16}$$

The parameter C_p depends on the spatial coherence of the input beam as

$$C_p = C_f\sqrt{1 + 2w_0^2/\sigma_c^2}, \qquad C_f = w_g^2/w_0^2, \tag{17}$$

where w_g is the spot size of the fundamental mode of the GRIN rod and is defined as $w_g = 1/\sqrt{kb}$. Periodic self-imaging of a partially coherent beam is evident from Eq. (15) and occurs at distances where $f(z)$ takes value 1. From Eq. (16), this happens when $bz = m\pi$, where m is an integer.

Equation (15) is remarkable because it shows that even a partially coherent Gaussian beam evolves in a self-similar fashion and maintains its Gaussian nature inside a GRIN rod. More specifically, the intensity of each spectral component of a partially coherent beam follows a periodic pattern that is similar to that of a coherent beam with one major difference: the parameter C_p is replaced with C_f for coherent beams. To understand the impact of partial coherence on a GRIN lens, we consider what happens at a distance $z = L_f = \pi/(2b)$ that corresponds to the the focal length of such a lens.

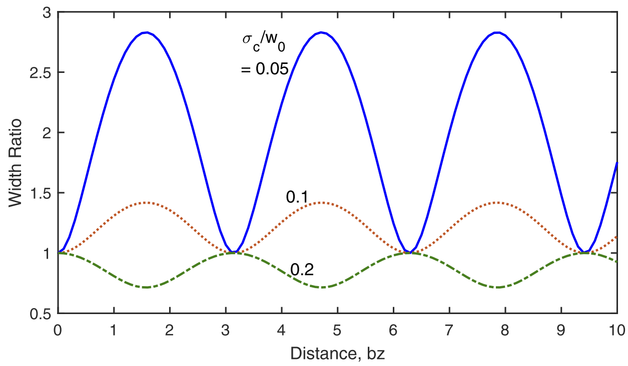

Fig. 1. Variations in the width of a partially coherent beam plotted as a function of bz for three values of the ratio σ_c/w_0.

The beam's partial coherence appears in Eq. (17) through the parameter σ_c. For a coherent beam, σ_c becomes much larger than the beam's width w_0, and C_p takes its minimum value $C_f = (w_g/w_0)^2$. We can estimate the value of the parameter w_g using

$$w_g^2 = \frac{1}{kb} = \frac{\lambda a}{2\pi n_0 \sqrt{2\Delta}}, \tag{18}$$

where λ is the input wavelength. Using $n_0 = 1.5$, $a = 0.5$ mm, and $\Delta = 0.02$ as typical values for a GRIN rod, w_g is about 16 μm at wavelengths near 1 μm. In practice, the condition $w_0 > w_g$ is satisfied as the width of the beam to be focused by the GRIN lens is likely to exceed 100 μm. It follows that the value of the parameter C_f is considerably smaller than 1.

As an example, Fig. 1 shows the width ratio w_e/w_0 as a function of bz for three values of σ_c/w_0 using $C_f = 0.1$. A coherent beam's width would be smaller by a factor of 10 in the first focal plane located at $bz = \pi/2$. Although some focusing of the beam occurs for $\sigma_c/w_0 = 0.2$, focusing ceases to occur when this ratio is close 0.1. As seen in Fig. 1, the beam expands by a factor of 2.8, rather than focusing, when $\sigma_c/w_0 = 0.05$. The incident beam is nearly incoherent for such small values of σ_c. We can think of such a beam as a superposition of multiple coherent beams that diffract over a wide region because of their tiny sizes. It is evident from these results that spatial coherence of the incoming beam affects considerably the focusing properties of a GRIN rod. Notice that a beam's width would not change at all inside the GRIN rod when its initial width is chosen to ensure $C_p = 1$.

Let us consider a GRIN lens whose length is chosen to be $L_f = \pi/(2b)$ so that a beam entering the lens comes to focus at the output end. It follows from Eqs. (16) and (17) that the spot size of the focused beam is given by

$$w_e(z = L_f) = \left(\frac{w_g^2}{w_0}\right)\sqrt{1 + 2w_0^2/\sigma_c^2}. \tag{19}$$

Figure 2 shows how the focal spot size varies with the ratio σ_c/w_0 for three values of input spot size w_0 using $w_g = 16$ μm for the GRIN lens. As expected, when the input beam is nearly coherent ($\sigma_c \geq w_0$), the focused beam acquires a small spot size (~ 1 μm) in the focal plane. The spot size increases for a partially coherent beam and can exceed 10 μm when the degree of spatial coherence of the input beam is such that $\sigma_c/w_0 = 0.1$. As we saw earlier, the beam does not compress at all and may even become wider at the focal point when σ_c is a small fraction of w_0.

Siegman introduced in 1990 a new quantity known as the M^2 factor [21]. It compares the spatial quality of an arbitrary beam

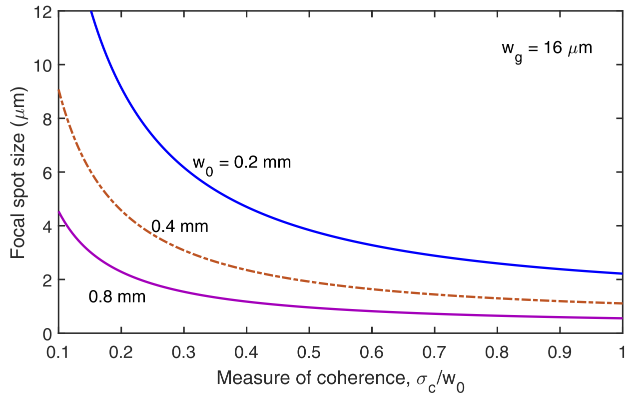

Fig. 2. Focal spot size of a partially coherent beam plotted as a function of σ_c/w_0 for three values of the beam's initial width.

with a coherent Gaussian beam and is based on the notion that the angular spread is minimum for a Gaussian beam at its waist. As the width of a beam becomes smaller at its waist, its angular spread becomes larger, as dictated by the angular spectrum of the beam. The M^2 factor compares the product of the spatial and angular widths of a beam with that of a Gaussian beam.

The concept of the M^2 factor has been applied to partially coherent beams [22]. In the case of the Gaussian–Schell model used here, the M^2 factor associated with such a partially coherent beam was found to be

$$M^2 = \sqrt{1 + 2w_0^2/\sigma_c^2}. \tag{20}$$

It follows that the compression factor in Eq. (19) can be written as $w_e/w_0 = C_f M^2$. These results show that the focusing ability of a partially coherent beam is degraded just by its M^2 factor. In other words, the focal spot size becomes wider by a factor of M^2 for a partially coherent beam.

In conclusion, we have used the standard coherence theory based on cross-spectral density to study how the focusing by a GRIN lens is affected when the incoming beam is partially coherent. The intensity of a partially coherent beam exhibits self-imaging and evolves in a periodic fashion in a GRIN medium with a parabolic index profile. Spatial coherence of the beam affects a single parameter that governs how much the beam is compressed at the focal point. Our results show that the focal spot size depends on the fraction of the beam's diameter over which coherence persists. Focusing ceases to occur, and the beam may even expand at the focal point of a GRIN lens, when this fraction is below 10%.

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Data availability. Data underlying the results presented in this paper are not publicly available at this time but may be obtained from the author upon reasonable request.

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