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Spatial beam narrowing in multimode graded-index fiber amplifiers: an analytic approach

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Doped and optically pumped graded-index (GRIN) fibers can be used to amplify an optical beam such that its spatial quality is improved at the output end of the fiber compared with that of the unamplified beam. We develop a simple model of the amplification process in such GRIN fiber amplifiers and show that the resulting equations can be solved analytically with suitable approximations. The solution shows that the width of the amplifying beam oscillates but also becomes narrower because of the radial dependence of the optical gain. The main advantage of our simplified approach is that it provides an analytic expression for the damping distance of beam-width oscillations that shows clearly the role played by various physical parameters.

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Considerable attention has been paid to nonlinear phenomena occurring in multimode graded-index (GRIN) fibers [1–4]. Recently, GRIN fibers have been doped with ytterbium and used as optical amplifiers [5–7]. It has been observed that the amplified signal's beam quality is better at the output end of the amplifier compared with that expected in the absence of amplification. This effect is similar to the Raman-induced beam cleanup observed in GRIN fibers for the Stokes beam generated through stimulated Raman scattering [8–11].

A mode-based description of the Raman beam cleanup process has shown that the use of GRIN fibers is essential for the improvement of the beam quality [9]. However, such an approach becomes less appropriate when many modes of the GRIN fiber are excited by the incoming pump and signal beams. Recent work on Kerr-induced beam cleaning has shown that periodic self-imaging [3], a unique property of GRIN fibers, plays an important role by creating a nonlinear index grating inside the GRIN fiber [12–14].

A nonmodal approach has recently been used for GRIN fiber amplifiers, to study spatial narrowing of the signal beam being amplified [7]. The resulting numerical model is time-consuming because it requires a solution of the coupled partial differential equations satisfied by the pump and signal beams, while also accounting for radial and axial variations of the optical gain, through the atomic rate equations. However, the model is quite comprehensive and is capable of including nonlinear effects, such as Kerr-induced beam cleaning.

In this work, we develop a simple model of the amplification process in GRIN fiber amplifiers. We show that the resulting equation for the signal being amplified can be solved analytically with suitable approximations, and that the solution provides considerable physical insight.

The article is organized as follows. We start with the mathematical details of the amplification process and discuss the approximations that can be used to develop a simplified model. Next, we solve the partial differential equation governing amplification of the signal and show that the beam's width satisfies an equation that is similar to the equation governing the behavior of a damped harmonic oscillator. We use this equation to discuss narrowing of the signal beam occurring during its amplification inside a GRIN fiber. The main results are summarized at the end of the article.

We consider a GRIN fiber with a nonuniform density of dopants in the radial direction. The fiber is cladding-pumped with an intense pump beam to invert the population of dopants and to amplify a signal beam propagating inside the GRIN fiber. Under such conditions, the optical gain varies radially but remains nearly uniform along the amplifier's length. The gain can be included through the imaginary part of the refractive index of the GRIN fiber, as

$$n(\rho, z) = n_0\left(1 - \frac{1}{2}b^2\rho^2\right) - i\,\frac{g(\rho, z)}{2n_0k_0}, \quad (1)$$

where $\rho = \sqrt{x^2 + y^2}$ is the radial distance from the central axis of the GRIN fiber (aligned with the z axis), n_0 is the refractive index at $\rho = 0$, b is the index gradient, $k_0 = \omega_0/c$, and ω_0 is the central frequency of the signal being amplified. The gain $g(\rho, z)$ depends on the local density of dopants and varies with ρ because of nonuniform doping of the GRIN fiber.

The signal incident on the GRIN fiber is in the form of a quasi-continuous beam with a spectrum centered at ω_0 and narrow enough that the effects of chromatic dispersion are negligible. In the absence of nonlinear effects, its electric field satisfies the Helmholtz equation:

$$\nabla^2\mathbf{E}_s + n^2k_0^2\mathbf{E}_s = 0. \quad (2)$$

We introduce the slowly varying amplitude A_s using $\mathbf{E}_s = \hat{\mathbf{p}}A_se^{ikz}$, where $\hat{\mathbf{p}}$ is the polarization unit vector, and make the paraxial approximation to obtain

$$2ik\frac{\partial A_s}{\partial z} + \nabla_T^2 A_s + (n^2k_0^2 - k^2)A_s = 0. \quad (3)$$

Here ∇_T^2 is the transverse part of the Laplacian operator. When we substitute $n(\rho, z)$ from Eq. (1), choose $k = n_0k_0$, and neglect several small terms, we obtain the following equation, governing the signal's amplification inside a GRIN fiber amplifier:

$$2ik\frac{\partial A_s}{\partial z} + \nabla_T^2 A_s - (kb\rho)^2 A_s - ikg(\rho, z)A_s = 0. \quad (4)$$

In this equation, the second, third, and fourth terms take into account, respectively, the effects of diffraction, the index gradient b, and the gain provided by dopants. The nonlinear effects are not included in Eq. (4), as it is assumed that the signal's power remains much smaller than that of the pump. As a result, Kerr-induced beam cleaning plays no role in our simplified model. The Kerr nonlinearity is known to produce self-cleaning of the pump beam [12–14]. Pump-induced cleaning of the signal might also occur through cross-phase modulation. We neglect it in this study to focus on the gain-induced spatial narrowing of the amplified signal.

Before solving Eq. (4), we need to specify the gain function $g(\rho, z)$. The local gain depends on both the density of the dopants and the intensity of the pump beam whose absorption creates population inversion. An accurate model of the gain $g(\rho, z)$ requires numerical solutions of the coupled pump and signal equations, together with the atomic rate equations to calculate the inversion density at any point inside the amplifier [7]. Our objective in this work is to develop a simple model that captures the essential physics of the problem and allows us to solve Eq. (4) analytically. We neglect gain saturation, assume that the amplifier is cladding-pumped, and ignore the z dependence of the gain. We also assume a parabolic form for the density of dopants and write $g(\rho, z)$ in the form

$$g(\rho, z) \approx g_0 - g_2\rho^2, \quad (5)$$

where g_0 and g_2 are constants. Physically, g_0 is the peak gain along the fiber's axis and g_2 depends on the distribution of the dopant's density in the radial direction.

We note that both g_0 and g_2 vary with z when the fiber has uniform doping and is pumped with a beam co-propagating with the signal [7]. We treat them as constants because this simplification allows us to solve Eq. (4) in an analytic fashion. We also assume that the signal beam enters the GRIN fiber as a collimated Gaussian beam and maintains its radial symmetry during amplification. With these simplifications, Eq. (4) takes the following form in cylindrical coordinates:

$$2ik\frac{\partial A_s}{\partial z} + \frac{\partial^2 A_s}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial A_s}{\partial \rho} - (kb\rho)^2 A_s = ik(g_0 - g_2\rho^2)A_s. \quad (6)$$

To solve Eq. (6), we assume that the signal maintains its Gaussian form but that its amplitude, width, phase-front curvature, and phase evolve along the amplifier's length. This allows us to write the solution of Eq. (6) in the form

$$A_s(\rho, z) = A_0 \exp\left(-\frac{\rho^2}{2w^2} + \frac{i}{2}kh\rho^2 + i\psi\right), \quad (7)$$

where the four parameters (A_0, w, h, and ψ) vary with z. Using this form in Eq. (6), we obtain the following equations for the four parameters:

$$\frac{dA_0}{dz} = \left(\tfrac{1}{2}g_0 - h\right)A_0, \quad (8)$$

$$\frac{d\psi}{dz} = -\frac{1}{kw^2}, \quad (9)$$

$$\frac{dw}{dz} = hw - \tfrac{1}{2}g_2w^3, \quad (10)$$

$$\frac{dh}{dz} = \frac{1}{k^2w^4} - h^2 - b^2. \quad (11)$$

Consider first the case of a constant gain ($g_2 = 0$). Equations (10) and (11) then become identical to those obtained for an undoped GRIN fiber. Taking the second derivative of the width w, we obtain

$$\frac{d^2w}{dz^2} + b^2w = \frac{1}{k^2w^3}. \quad (12)$$

This equation is easily solved, and the solution for the beam's width is found to be [2]

$$w(z) = w_s\sqrt{f(z)}, \qquad f(z) = 1 - (1 - C_s^2)\sin^2(bz), \quad (13)$$

where $C_s = (w_g/w_s)^2$, $w_g = (kb)^{-1/2}$, and w_s is the input width of the beam being amplified. This result shows that the beam's width varies in a periodic fashion because of self-imaging in GRIN fibers [3]. The parameter w_g is related to the width of the fundamental mode of the GRIN fiber. The parameter C_s depends on the input width of the beam and satisfies the condition $C_s<1$, as $w_s>w_g$ in practice. Physically, C_s is the fraction by which the signal's width is reduced in the focal plane during each self-imaging period.

Because of the optical gain g_0, the beam's power increases with distance. We can see this feature by using $dw/dz = hw$ in Eq. (8) and writing it as

$$\frac{d}{dz}(w^2A_0^2) = g_0(w^2A_0^2). \quad (14)$$

Recalling that the signal's power at any distance z is given by

$$P(z) = \int_0^{2\pi} d\phi \int_0^{\infty} |A_s(\rho, z)|^2 \rho d\rho = (\pi w^2)A_0^2, \quad (15)$$

we obtain $dP/dz = g_0P$, with the solution $P(z) = P(0)\exp(g_0z)$. As expected, the beam's power increases exponentially inside the amplifier.

The situation is considerably different when the gain is radially nonuniform and g_2 is finite in Eq. (5). To the first order in g_2, the width equation in Eq. (12) has an additional term and becomes

$$\frac{d^2w}{dz^2} + b^2w = \frac{1}{k^2w^3} - 2g_2w^2\frac{dw}{dz}. \quad (16)$$

The last term in this equation governs the effects of a spatially nonuniform gain in the transverse dimensions. Its presence modifies the periodic solution given in Eq. (13). On physical grounds, we expect the amplitude of periodic variations to decrease with increasing z because of a larger gain in the central region of the signal beam compared with its outer regions. In other words, the beam should become narrower as the signal is amplified inside such a fiber amplifier.

Before solving Eq. (16) numerically, we normalize it using $s = w/w_s$ and $\xi = bz$. The resulting equation for the normalized

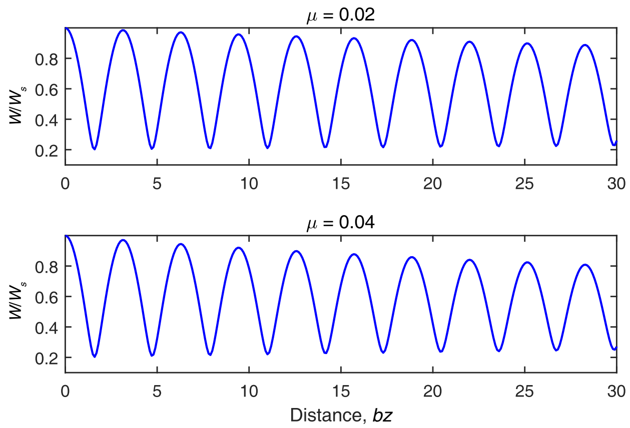

Fig. 1. Oscillations in the width of the signal beam for two values of the parameter μ. The amplitude of oscillations does not change when $\mu = 0$.

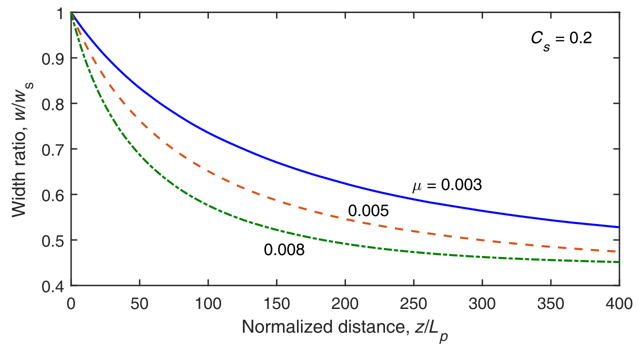

Fig. 2. Predicted decrease in the width of the signal beam as a function of z/L_p for three values of μ.

width s is

$$\frac{d^2s}{d\xi^2} + s = \frac{C_s^2}{s^3} - 2\mu s^2 \frac{ds}{dz}, \tag{17}$$

where $\mu = (g_2/b)w_s^2$ is a dimensionless parameter, such that $\mu \ll 1$. Figure 1 shows the results obtained by solving Eq. (17) using $C_s = 0.2$, with $\mu = 0.02$ (top) or $\mu = 0.04$ (bottom). The width oscillates because of self-imaging but the amplitude of oscillations decreases with distance. The reduction in amplitude does not occur for $\mu = 0$, and the damping rate of oscillations increases with μ.

The important question is what values of μ are realistic for a GRIN fiber amplifier. It is useful to introduce the self-imaging period of a GRIN fiber, as $L_p = 2\pi/b$. If we assume that the gain in Eq. (5) is reduced by a factor of 4 at $\rho = a$, we can use $g_2 = g_0/(2a^2)$ and write μ as

$$\mu = (3g_0 L_p/8\pi)(w_s/a)^2. \tag{18}$$

For a GRIN fiber providing 20 dB amplification over its 10 m length, g_0 is found to be $0.46\,\mathrm{m}^{-1}$ from the relation $e^{gL} = 100$. The self-imaging period L_p of a GRIN fiber is close to 1 mm. If we use $w_s = 15\,\mu\mathrm{m}$ with $a = 25\,\mu\mathrm{m}$, the estimated value of μ is about 2×10^{-5}. Such low values of μ indicate that the decrease in the amplitude of periodic oscillations would become apparent in Fig. 1 only after thousands of oscillations in Fig. 1.

We solve Eq. (16) approximately to estimate such a slow damping rate of oscillations. When $g_2 = 0$, the solution is given in Eq. (13). We use this solution to replace w^2 in the g_2 term with $w_s^2\langle f(z)\rangle$, where the average is over one self-imaging period. This allows us to write Eq. (16) as

$$\frac{d^2w}{dz^2} + 2\gamma_d \frac{dw}{dz} + b^2 w = \frac{1}{k^2 w^3}, \tag{19}$$

where $\gamma_d = g_2 w_s^2 \langle f(z)\rangle$. This equation corresponds to a damped harmonic oscillator when the term on the right is negligible. Its form suggests that an approximate solution of Eq. (16) is given by

$$w(z) = w_s\sqrt{f(z)}\exp(-\gamma_d z). \tag{20}$$

Its physical meaning is clear. The beam's width oscillates with the self-imaging period, as expected for a GRIN medium, but the amplitude of oscillations decreases with z, resulting in spatial narrowing of the signal beam inside the fiber amplifier.

Figure 2 shows the predicted decrease in the width of a signal beam over 400 self-imaging periods for three values of μ, using $C_s = 0.2$. These numerical simulations agree with the exponential decay predicted by Eq. (20). The maximum value of the width at the end of each self-imaging period is plotted to show the envelope without any oscillations. Relatively large values of μ were used to reduce the computing time, but the same qualitative behavior is expected for smaller values of μ.

The distance within the amplifier over which the width is reduced by a factor of $1/e$ is given by

$$L_d = 1/\gamma_d = \left[\tfrac{1}{2} g_2 w_s^2 (1 + C_s^2)\right]^{-1}, \tag{21}$$

where we calculated $\langle f(z)\rangle$ from Eq. (13). The parameter C_s depends on the width of input beam and has a value of about 0.13 for $w_s = 15\,\mu\mathrm{m}$. Using $g_2 = 3g_0/(4a^2)$ with $g_0 = 0.46\,\mathrm{m}^{-1}$ and $a = 25\,\mu\mathrm{m}$, L_d is estimated to be 15 m. Thus, considerable narrowing of the signal beam can occur over the 10 m length of the amplifier. If the signal beam's width is reduced from its initial value of 15 μm to 7 μm, most of its power at the output end will be in the fundamental mode of the GRIN fiber amplifier. This conclusion agrees with a detailed numerical model that includes gain saturation and many other effects by solving the pump and signal equations together [7].

We briefly discuss how our model can be used for uniformly doped GRIN fibers, pumped with an intense pump beam that co-propagates with the signal being amplified. Under such conditions, as mentioned earlier, the gain parameters g_0 and g_2 vary with z along the amplifier's length. The z dependence of g_0 affects the signal's power but has no effect on its width. The width's equation in Eq. (16) has an extra term that depends on the derivative dg_2/dz as

$$\frac{d^2w}{dz^2} + b^2 w = \frac{1}{k^2 w^3} - 2g_2(z) w^2 \frac{dw}{dz} - \frac{1}{2}\frac{dg_2}{dz} w^3. \tag{22}$$

If the z dependence of g_2 is known, one can solve Eq. (22) numerically to study how the signal's width is reduced over the amplifier's length.

In conclusion, a simple analytic model is presented that allows us to study beam narrowing in multimode GRIN fiber amplifiers. It shows that the signal beam narrows as it is amplified on a length scale ∼1 m, while also exhibiting periodic self-imaging on a length scale ∼1 mm. The predicted beam narrowing has its origin in the radial dependence of optical gain. The main advantage of our simplified approach is that it provides an analytic expression for the damping rate of oscillations, which shows clearly the role played by various physical parameters.

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Data availability. Data underlying the results presented in this paper are not publicly available at this time but may be obtained from the author upon reasonable request.

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