Optical Communication Systems (OPT428)

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Chapter 3:
Signal Propagation in Optical Fibers

- Fiber fundamentals
- Basic propagation equation
- Impact of fiber losses
- Impact of fiber dispersion
- Polarization-mode dispersion
- Polarization-dependent losses

Optical Fibers

- Most suitable as communication channel because of dielectric waveguiding (act like an optical wire).
- Total internal reflection at the core-cladding interface.
- Single-mode propagation for core size $< 10\ \mu$m.

What happens to Signal?

- Fiber losses: limit the transmission distance (minimum loss near 1.55 μm).
- Chromatic dispersion: limits the bit rate through pulse broadening.
- Nonlinear effects: distort the signal and limit the system performance.

Single-Mode Fibers

- Fibers support only one mode when the core size is such that $V = k_0 a\sqrt{n_1^2 - n_2^2} < 2.405$.
- This mode is almost linearly polarized $(|E_z|^2 \ll |E_T|^2)$.
- Spatial mode distribution approximately Gaussian

$$E_x(x,y,z,\omega) = A_0(\omega)\exp\left(-\frac{x^2+y^2}{w^2}\right)\exp[i\beta(\omega)z].$$

- Spot size: $w/a \approx 0.65 + 1.619V^{-3/2} + 2.879V^{-6}$.
- Mode index: $\bar{n} = n_2 + b(n_1 - n_2) \approx n_2(1 + b\Delta)$

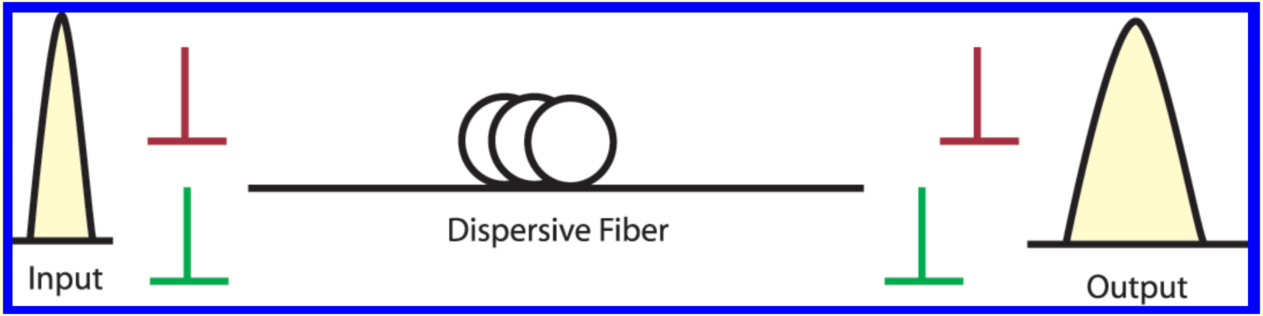
$$b(V) \approx (1.1428 - 0.9960/V)^2.$$

Fiber Dispersion

Origin: Frequency dependence of the mode index $n(\omega)$:

$$\beta(\omega) = \bar{n}(\omega)\omega/c = \beta_0 + \beta_1(\omega - \omega_0) + \beta_2(\omega - \omega_0)^2 + \cdots .$$

- Transit time for a fiber of length L: $T = L/v_g = \beta_1 L$.
- Different frequency components travel at different speeds and arrive at different times at output end (pulse broadening).



Fiber Dispersion (continued)

- Pulse broadening is governed by group-velocity dispersion.
- Consider delay in arrival time over a bandwidth $\Delta\omega$:

$$\Delta T = \frac{dT}{d\omega}\Delta\omega = \frac{d}{d\omega}\frac{L}{v_g}\Delta\omega = L\frac{d\beta_1}{d\omega}\Delta\omega = L\beta_2\Delta\omega.$$

- $\Delta\omega$ is pulse bandwidth and L is fiber length.
- GVD parameter: $\beta_2 = \left(\frac{d^2\beta}{d\omega^2}\right)_{\omega=\omega_0}$.
- Alternate definition: $D = \frac{d}{d\lambda}\left(\frac{1}{v_g}\right) = -\frac{2\pi c}{\lambda^2}\beta_2$.
- Limitation on the bit rate: $\Delta T < T_b = 1/B$, or

$$B(\Delta T) = BL\beta_2\Delta\omega \equiv BLD\Delta\lambda < 1.$$

Basic Propagation Equation

- Optical field inside a single-mode fiber has the form

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}[\hat{\mathbf{e}} F(x,y) A(z,t) \exp(i\beta_0 z - i\omega_0 t)].$$

- $F(x,y)$ represents spatial profile of the fiber mode.
- $\hat{\mathbf{e}}$ is the polarization unit vector.
- Since pulse amplitude $A(z,t)$ does not depend on x and y, we need to solve a simple one-dimensional problem.
- It reduces to a scalar problem if $\hat{\mathbf{e}}$ does not change with z. Let us assume this to be the case.
- It is useful to work in the frequency domain $(\Delta\omega = \omega - \omega_0)$:

$$A(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}(z,\omega) \exp(-i\Delta\omega t)\, d(\Delta\omega).$$

Spectral Amplitude

- Consider one frequency component at ω. Its amplitude at z is related to that at $z = 0$ by a simple phase factor:

$$\tilde{A}(z, \omega) = \tilde{A}(0, \omega) \exp[i\beta_p(\omega)z - i\beta_0 z].$$

- Propagation constant has the following general form:

$$\beta_p(\omega) \approx \beta_L(\omega) + \beta_{NL}(\omega_0) + i\alpha(\omega_0)/2.$$

- $\beta_L(\omega) = \bar{n}(\omega)\omega/c, \quad \beta_{NL}(\omega_0) = (\omega_0/c)\delta n_{NL}(\omega_0).$
- Expand $\beta_L(\omega)$ is a Taylor series around ω_0:

$$\beta_L(\omega) \approx \beta_0 + \beta_1(\Delta\omega) + \frac{\beta_2}{2}(\Delta\omega)^2 + \frac{\beta_3}{6}(\Delta\omega)^3,$$

where $\beta_m = (d^m\beta/d\omega^m)_{\omega=\omega_0}$.

Dispersion Parameters

- Parameter β_1 is related inversely to group velocity: $\beta_1 = 1/v_g$.
- β_2 and β_3: second- and third-order dispersion parameters.
- They are responsible for pulse broadening in optical fibers.
- β_2 is related to D as $D = \frac{d}{d\lambda}\left(\frac{1}{v_g}\right) = -\frac{2\pi c}{\lambda^2}\beta_2$.
- It vanishes at the zero-dispersion wavelength ($\lambda_{\rm ZD}$).
- Near this wavelength, D varies linearly as $D \approx S(\lambda - \lambda_{\rm ZD})$.
- β_3 is related to the dispersion slope S as $S = (2\pi c/\lambda^2)^2\beta_3$.
- Parameters $\lambda_{\rm ZD}$, D, and S vary from fiber to fiber.
- Fibers with relatively small values of D in the spectral region near 1.55 μm are called dispersion-shifted fibers.

Dispersive Characteristics of Some Commercial Fibers

Fiber Type and Trade Name	A_{eff} (μm^2)	λ_{ZD} (nm)	D (C band) ps/(km-nm)	Slope S ps/(km-nm^2)
Corning SMF-28	80	1302–1322	16 to 19	0.090
Lucent AllWave	80	1300–1322	17 to 20	0.088
Alcatel ColorLock	80	1300–1320	16 to 19	0.090
Corning Vascade	101	1300–1310	18 to 20	0.060
TrueWave-RS	50	1470–1490	2.6 to 6	0.050
Corning LEAF	72	1490–1500	2 to 6	0.060
TrueWave-XL	72	1570–1580	−1.4 to −4.6	0.112
Alcatel TeraLight	65	1440–1450	5.5 to 10	0.058

- Effective mode area A_{eff} of a fiber depends on $F(x,y)$.
- $A_{\text{eff}} = \pi w^2$ if mode shape is approximated with a Gaussian.

Pulse Propagation Equation

- Pulse envelope is obtained using

$$A(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{A}(0,\Delta\omega)\exp[i\beta_p(\omega)z - i\beta_0 z]d(\Delta\omega).$$

- Substitute $\beta_p(\omega)$ in terms of its Taylor expansion.
- Calculate $\partial A/\partial z$ and convert to time domain by replacing $\Delta\omega$ with $i(\partial A/\partial t)$.
- Final equation:

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} - \frac{\beta_3}{6}\frac{\partial^3 A}{\partial t^3} = i\beta_{NL}A - \frac{\alpha}{2}A.$$

Nonlinear Contribution

- Nonlinear term van be written as

$$\beta_{NL} = \frac{2\pi}{\lambda_0}(n_2 I) = \frac{2\pi n_2}{\lambda_0}\frac{|A|^2}{A_{\text{eff}}} = \gamma |A|^2.$$

- Nonlinear parameter $\gamma = \frac{2\pi n_2}{\lambda_0 A_{\text{eff}}}$.
- For pure silica $n_2 = 2.6 \times 10^{-20}$ m^2/W.
- Effective mode area A_{eff} depends on fiber design and varies in the range 50–80 μm^2 for most fibers.
- As an example, $\gamma \approx 2.1$ W^{-1}/km for a fiber with $A_{\text{eff}} = 50$ μm^2.
- Fibers with a relatively large value of A_{eff} are called large-effective-area fibers (LEAFs).

Pulse Propagation Equation

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} - \frac{\beta_3}{6}\frac{\partial^3 A}{\partial t^3} = i\gamma |A|^2 A - \frac{\alpha}{2} A.$$

- β_1 term corresponds to a constant delay experienced by a pulse as it propagates through the fiber.
- Since this delay does not affect the signal quality in any way, it is useful to work in a reference frame moving with the pulse.
- This can be accomplished by introducing new variables t' and z' as $t' = t - \beta_1 z$ and $z' = z$:

$$\frac{\partial A}{\partial z'} + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t'^2} - \frac{\beta_3}{6}\frac{\partial^3 A}{\partial t'^3} = i\gamma |A|^2 A - \frac{\alpha}{2} A.$$

Nonlinear Schrödinger Equation

- Third-order dispersive effects are often negligible in practice.
- Setting $\beta_3 = 0$, we obtain

$$\frac{\partial A}{\partial z} + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} = i\gamma|A|^2 A - \frac{\alpha}{2}A.$$

- If we also neglect losses and set $\alpha = 0$, we obtain the so-called NLS equation

$$i\frac{\partial A}{\partial z} - \frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2} + i\gamma|A|^2 A = 0.$$

- If we neglect the nonlinear term as well (low-power case)

$$i\frac{\partial A}{\partial z} - \frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2} = 0.$$

Linear Low-Power Case

- Pulse propagation in a fiber is governed by

$$i\frac{\partial A}{\partial z} - \frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2} = 0.$$

- Compare it with the paraxial equation governing diffraction:

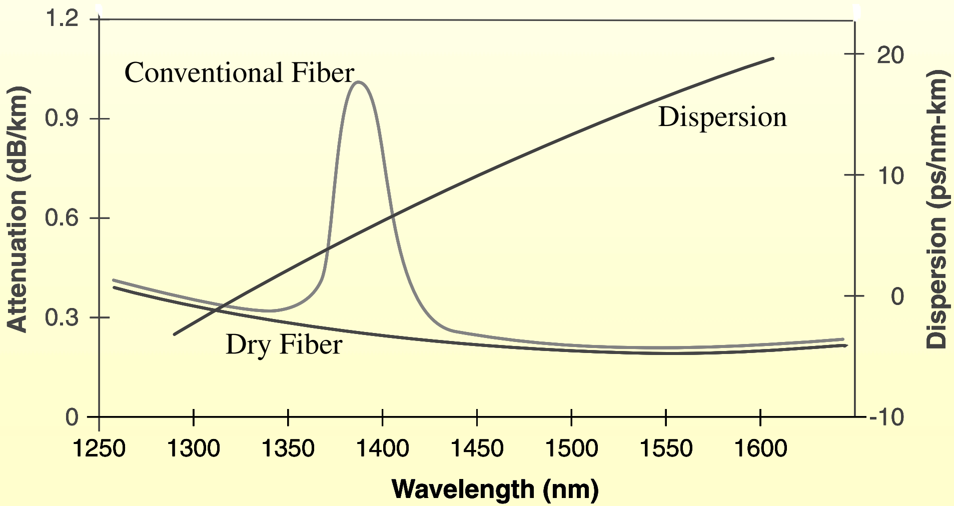
$$2ik\frac{\partial A}{\partial z} + \frac{\partial^2 A}{\partial x^2} = 0.$$

- Slit-diffraction problem identical to pulse propagation problem.
- The only difference is that β_2 can be positive or negative.
- Many results from diffraction theory can be used for pulses.
- A Gaussian pulse should spread but remain Gaussian in shape.

Fiber Losses

Definition: $\alpha(\mathrm{dB/km}) = -\frac{10}{L}\log_{10}\left(\frac{P_{\mathrm{out}}}{P_{\mathrm{in}}}\right) \approx 4.343\alpha.$

- Material absorption (silica, impurities, dopants)
- Rayleigh scattering (varies as λ^{-4})
- Waveguide imperfections (macro and microbending)



Impact of Fiber Losses

- Fiber losses reduce signal power inside the fiber.
- They also reduce the strength of nonlinear effects.
- Using $A(z,t) = B(z,t)\exp(-\alpha z/2)$ in the NLS equation, we obtain

$$\frac{\partial B}{\partial z} + \frac{i\beta_2}{2}\frac{\partial^2 B}{\partial t^2} = i\gamma e^{-\alpha z}|B|^2 B.$$

- Optical power $|A(z,t)|^2$ decreases as $e^{-\alpha z}$ because of fiber losses.
- Decrease in the signal power makes nonlinear effects weaker, as expected intuitively.
- Loss is quantified in terms of the average power defined as

$$P_{\rm av}(z) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|A(z,t)|^2\,dt = P_{\rm av}(0)e^{-\alpha z}.$$

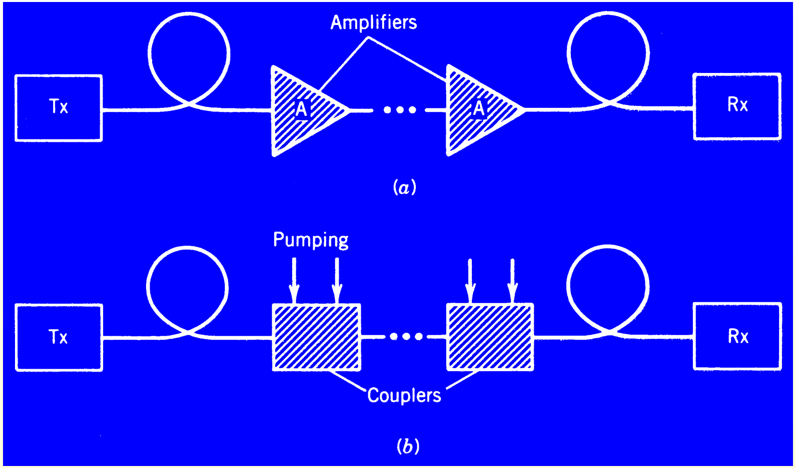
Loss Compensation

- Fiber losses must be compensated for distances >100 km.
- A *repeater* can be used for this purpose.
- A repeater is a receiver–transmitter pair: Receiver output is directly fed into an optical transmitter.
- Optical bit stream first converted into electric domain.
- It is then regenerated with the help of an optical transmitter.
- This technique becomes quite cumbersome and expensive for WDM systems as it requires demultiplexing of individual channels at each repeater.
- Alternative Solution: Use optical amplifiers.

Optical Amplifiers

- Several kinds of optical amplifiers were developed in the 1980s to solve the loss problem.
- Examples include: semiconductor optical amplifiers, Raman amplifiers, and erbium-doped fiber amplifiers (EDFAs).
- They amplify multiple WDM channels simultaneously and thus are much more cost-effective.
- All modern WDM systems employ optical amplifiers.
- Amplifier can be cascaded and thus enable one to transmit over distances as long as 10,000 km.
- We can divide amplifiers into two categories known as lumped and distributed amplifiers.

Lumped and Distributed Amplifiers



- In lumped amplifiers losses accumulated over 70–80 km are compensated using short lengths (∼10 m) of EDFAs.
- Distributed amplification uses the transmission fiber itself for amplification through stimulated Raman scattering.
- Pump light injected periodically using directional couplers.

Amplifier Noise

- All amplifiers degrade the SNR of an optical bit stream.
- They add noise to the signal through spontaneous emission.
- Noise can be included by adding a noise term to the NLS equation:

$$\frac{\partial A}{\partial z} + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} = i\gamma|A|^2 A + \frac{1}{2}[g_0(z) - \alpha]A + n(z,t).$$

- $g_0(z)$ is the gain coefficient of amplifiers used.
- Langevin noise $n(z,t)$ accounts for amplifier noise.
- Noise vanishes on average, i.e., $\langle n(z,t)\rangle = 0$.
- Noise is assumed to be Gaussian with the second moment:

$$\langle n(z,t)n(z',t')\rangle = g_0(z)h\nu\delta(z-z')\delta(t-t').$$

Lumped Amplification

- Length of each amplifier is much shorter ($l_a < 0.1$ km) than spacing L_A between two amplifiers (70–80 km).
- In each fiber section of length L_A, $g_0 = 0$ everywhere except within each amplifier of length l_a.
- Losses reduce the average power by a factor of $\exp(\alpha L_A)$.
- They can be fully compensated if gain $G_A = \exp(g_0 l_a) = \exp(\alpha L_A)$.
- EDFAs are inserted periodically and their gain is adjusted such that $G_A = \exp(\alpha L_A)$.
- It is not necessary that amplifier spacing be uniform.
- In the case of nonuniform spacing, $G_n = \exp[\alpha(L_n - L_{n-1})]$.

Distributed Amplification

- NLS equation with gain is solved for the entire fiber link.
- $g_0(z)$ chosen such that losses are fully compensated.
- Let $A(z,t) = \sqrt{p(z)}B(z,t)$, where $p(z)$ governs power variations along the link length.
- $p(z)$ is found to satisfy $\frac{dp}{dz} = [g_0(z) - \alpha]p$.
- If $g_0(z) = \alpha$ for all z, fiber is effectively lossless.
- In practice, pump power does not remain constant because of fiber losses at the pump wavelength.
- Losses can still be compensated over a distance L_A if the condition $\int_0^{L_A} g_0(z)\,dz = \alpha L_A$ is satisfied.
- Distance L_A is called the *pump-station spacing*.

Distributed Raman Amplification

- Stimulated Raman scattering is used for distributed amplification.
- Scheme works by launching CW power at several wavelengths at pump stations spaced apart 80 to 100 km.
- Wavelengths of pump lasers are chosen near 1.45 μm for amplifying 1.55-μm WDM signals.
- Wavelengths and pump powers are chosen to provide a uniform gain over the entire C band (or C and L bands).
- Backward pumping is commonly used for distributed Raman amplification.
- Such a configuration minimizes the transfer of pump intensity noise to the amplified signal.

Bidirectional Pumping

- Use of a bidirectional pumping scheme beneficial in some cases.
- Consider when one pump laser is used at both ends of a fiber section of length L_A. Gain coefficient $g(z)$ is of the form

$$g_0(z) = g_1 \exp(-\alpha_p z) + g_2 \exp[-\alpha_p (L_A - z)].$$

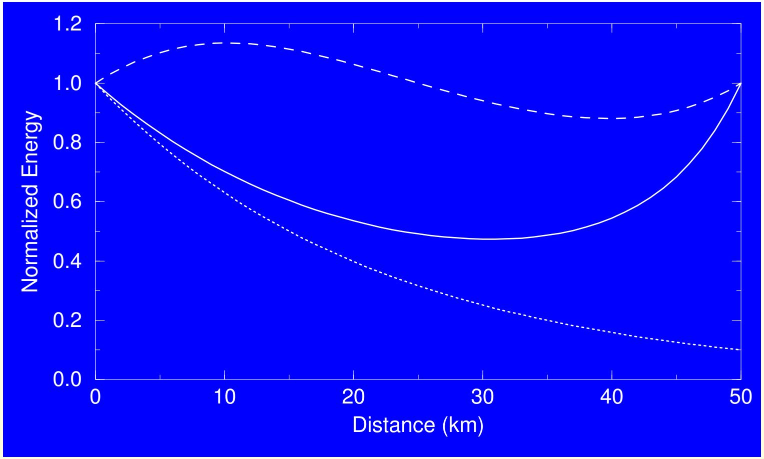
- Integrating $\frac{dp}{dz} = [g_0(z) - \alpha]p$, average power is found to vary as

$$p(z) = \exp\left[\alpha L_A \left(\frac{\sinh[\alpha_p(z - L_A/2)] + \sinh(\alpha_p L_A/2)}{2\sinh(\alpha_p L_A/2)}\right) - \alpha z\right].$$

- In the case of backward pumping, $g_1 = 0$, and

$$p(z) = \exp\left\{\alpha L_A \left[\frac{\exp(\alpha_p z) - 1}{\exp(\alpha_p L_A) - 1}\right] - \alpha z\right\}.$$

Bidirectional Pumping



- Variations in signal power between two pump stations.
- Backward pumping: (solid line). Bidirectional pumping (dashed line). Lumped amplifiers: dotted line. $L_A = 50$ km in all cases.
- Signal power varies by a factor of 10 in lumped case, by a factor <2 for backward pumping and <15% for bidirectional pumping.

Impact of Fiber Dispersion

- Different spectral components of the signal travel at slightly different velocities within the fiber.
- This phenomenon is referred to as group-velocity dispersion (GVD).
- GVD parameter β_2 governs the strength of dispersive effects.
- How GVD limits the performance of lightwave systems?
- To answer this question, neglect the nonlinear effects ($\gamma = 0$).
- Assuming fiber losses are compensated periodically ($\alpha = 0$), dispersive effects are governed by

$$\frac{\partial A}{\partial z} + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} = 0.$$

General Solution

- Using the Fourier-transform method, general solution is given by

$$A(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{A}(0,\omega)\exp\left(\frac{i}{2}\beta_2 z\omega^2 - i\omega t\right) d\omega.$$

- $\tilde{A}(0,\omega)$ is the Fourier transform of $A(0,t)$:

$$\tilde{A}(0,\omega) = \int_{-\infty}^{\infty} A(0,t)\exp(i\omega t)\,dt.$$

- It follows from the linear nature of problem that we can study dispersive effects for individual pulses in an optical bit stream..
- Considerable insight is gained by focusing on the case of chirped Gaussian pulses.

Chirped Gaussian Pulses

- Chirped Gaussian Pulse at $z = 0$:

$$A(0,t) = A_0 \exp\left[-\frac{(1+iC)t^2}{2T_0^2}\right].$$

- Input pulse width: $T_{\mathrm{FWHM}} = 2(\ln 2)^{1/2} T_0 \approx 1.665 T_0$.
- Input chirp: $\delta\omega(t) = -\frac{\partial \phi}{\partial t} = \frac{C}{T_0^2} t$.
- Pulse spectrum:

$$\tilde{A}(0,\omega) = A_0 \left(\frac{2\pi T_0^2}{1+iC}\right)^{1/2} \exp\left[-\frac{\omega^2 T_0^2}{2(1+iC)}\right].$$

- Spectral width: $\Delta\omega_0 = \sqrt{1+C^2}/T_0$.

Pulse Broadening

- Optical field at a distance z is found to be:

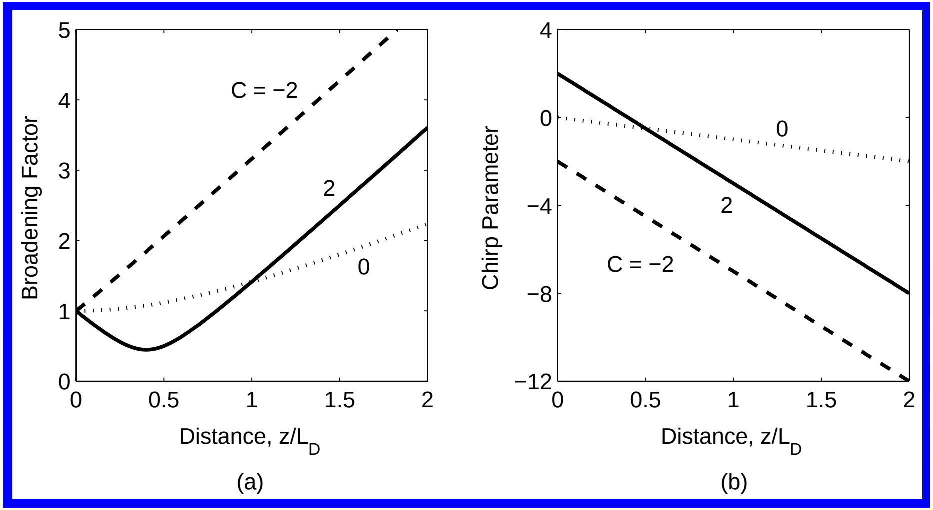
$$A(\xi,t) = \frac{A_0}{\sqrt{b_f}} \exp\left[-\frac{(1+iC_1)t^2}{2T_0^2 b_f^2} + \frac{i}{2}\tan^{-1}\left(\frac{\xi}{1+C\xi}\right)\right].$$

- Normalized distance $\xi = z/L_D$ is introduced through the *dispersion length* defined as $L_D = T_0^2/|\beta_2|$.
- Broadening factor b_f and chirp C_1 vary with ξ as

$$b_f(\xi) = [(1+sC\xi)^2 + \xi^2]^{1/2}, \qquad C_1(\xi) = C + s(1+C^2)\xi.$$

- $s = \mathrm{sgn}(\beta_2) = \pm 1$ depending on whether pulse propagates in the normal or anomalous dispersion region of the fiber.
- Main Conclusion: Pulse maintains its Gaussian shape but its width and chirp change with propagation.

Evolution of Width and Chirp



$$\frac{T_1}{T_0} = \left[\left(1 + \frac{C\beta_2 z}{T_0^2}\right)^2 + \left(\frac{\beta_2 z}{T_0^2}\right)^2\right]^{1/2}.$$

- Broadening depends on the sign of $\beta_2 C$.
- Unchirped pulse broadens by a factor $\sqrt{1+(z/L_D)^2}$.

Effect of Chirp

- When $\beta_2 C > 0$, chirped pulse broadens more because dispersion-induced chirp adds to the input chirp.
- If $\beta_2 C < 0$, dispersion-induced chirp is of opposite kind.
- From $C_1(\xi) = C + s(1 + C^2)\xi$, C_1 becomes zero at a distance $\xi = |C|/(1 + C^2)$, and pulse becomes unchirped.
- Pulse width becomes minimum at that distance:

$$T_1^{\min} = T_0/\sqrt{1 + C^2}.$$

- Pulse rebroadens beyond this point and its width eventually becomes larger than the input width.
- Chirping in time for a pulse is analogous to curvature of the wavefront for an optical beam.

Pulses of Arbitrary Shape

- In practice, pulse shape differs from a Gaussian.
- Third-order dispersion effects become important close to the zero-dispersion wavelength.
- A Gaussian pulse does not remains Gaussian in shape when effects of β_3 are included.
- Such pulses cannot be properly characterized by their FWHM.
- A measure of pulse width for pulses of arbitrary shapes is the root-mean square (RMS) width:

$$\sigma_p^2(z) = \langle t^2 \rangle - \langle t \rangle^2, \qquad \langle t^m \rangle = \frac{\int_{-\infty}^{\infty} t^m |A(z,t)|^2 \, dt}{\int_{-\infty}^{\infty} |A(z,t)|^2 \, dt}.$$

RMS Pulse Width

- It turns out that σ_p can be calculated for pulses of arbitrary shape, while including dispersive effects to all orders.
- Derivation is based on the observation that pulse spectrum does not change in a linear dispersive medium.
- First step consists of expressing the moments in terms of $\tilde{A}(z,\omega)$:

$$\langle t \rangle = \frac{-i}{N}\int_{-\infty}^{\infty} \tilde{A}^* \frac{\partial \tilde{A}}{\partial \omega} d\omega, \qquad \langle t^2 \rangle = \frac{1}{N}\int_{-\infty}^{\infty} \left|\frac{\partial \tilde{A}}{\partial \omega}\right|^2 d\omega.$$

- $N \equiv \int_{-\infty}^{\infty} |\tilde{A}(z,\omega)|^2 d\omega$ is a normalization factor.
- Spectral components propagate according to the simple relation

$$\tilde{A}(z,\omega) = \tilde{A}(0,\omega)\exp[i\beta_L(\omega)z - i\beta_0 z].$$

- $\beta_L(\omega)$ includes dispersive effects to all orders.

RMS Pulse Width

- Let $\tilde{A}(0,\omega) = S(\omega)e^{i\theta}$; $\theta(\omega)$ related to chirp.
- RMS width σ_p at $z = L$ is found from the relation

$$\sigma_p^2(L) = \sigma_0^2 + [\langle\tau^2\rangle - \langle\tau\rangle^2] + 2[\langle\tau\theta_\omega\rangle - \langle\tau\rangle\langle\theta_\omega\rangle].$$

- $\langle f\rangle = \frac{1}{N}\int_{-\infty}^{\infty} f(\omega)S^2(\omega)\,d\omega$; $\theta_\omega = d\theta/d\omega$.
- Group delay τ is defined as $\tau(\omega) = (d\beta_L/d\omega)L$.
- Conclusion: $\sigma_p^2(L)$ is a quadratic polynomial of L.
- Broadening factor can be written as

$$f_b = (1 + c_1L + c_2L^2)^{1/2}.$$

- This form applies for pulses of any shape propagating inside a fiber link with arbitrary dispersion characteristics.

Rect-Shape Pulses

- For a pulse of width $2T_0$, $A(0,t) = A_0$ for $|t| < T_0$.
- Taking the Fourier transform:

$$S(\omega) = (2A_0T_0)\,\mathrm{sinc}(\omega T_0), \qquad \theta(\omega) = 0.$$

- Expanding $\beta_L(\omega)$ to second-order in ω, group delay is given by $\tau(\omega) = (d\beta_L/d\omega)L = (\beta_1 + \beta_2\omega)L$.
- We can now calculate all averaged quantities:

$$\langle\tau\rangle = \beta_1 L, \quad \langle\tau^2\rangle = \beta_1^2 + \beta_2^2 L^2/2T_0^2, \quad \langle\tau\theta_\omega\rangle = 0, \quad \langle\theta_\omega\rangle = 0.$$

- Final result for the RMS width is found to be

$$\sigma_p^2(L) = \sigma_0^2 + \tfrac{1}{2}T_0^2\xi^2 = \sigma_0^2(1 + \tfrac{3}{2}\xi^2).$$

- Rectangular pulse broadens more than a Gaussian pulse under the same conditions because of its sharper edges.

Hyperbolic Secant Pulses

- For an unchirped pulse, $A(0,t) = A_0 \text{sech}(t/T_0)$.
- Such pulses are relevant for soliton-based systems.
- Taking the Fourier transform:

$$S(\omega) = (\pi A_0 T_0)\,\text{sech}(\pi\omega T_0/2), \qquad \theta(\omega) = 0.$$

- Using $\tau(\omega) = (\beta_1 + \beta_2\omega)L$ and $\int_{-\infty}^{\infty} x^2 \text{sech}^2(x)\,dx = \pi^2/6$

$$\langle\tau\rangle = \beta_1 L, \qquad \langle\tau^2\rangle = \beta_1^2 + \pi^2\beta_2^2 L^2/(12T_0^2),$$

- RMS width of sech pulses increases with distance as

$$\sigma_p^2(L) = \sigma_0^2(1 + \tfrac{1}{2}\xi^2), \qquad \sigma_0^2 = \pi^2 T_0^2/6.$$

- A sech pulse broaden less than a Gaussian pulse because its tails decay slower than a Gaussian pulse.

Effects of Third-Order Dispersion

- Consider the effects of β_3 on Gaussian pulses.
- Expanding $\beta_L(\omega)$ to third-order in ω, group delay is given by

$$\tau(\omega) = (\beta_1 + \beta_2\omega + \tfrac{1}{2}\beta_3\omega^2)L.$$

- Taking Fourier transform of $A(0,t)$, we obtain

$$S^2(\omega) = \frac{4\pi A_0^2\sigma_0^2}{1+C^2}\exp\left(-\frac{2\omega^2\sigma_0^2}{1+C^2}\right), \quad \theta(\omega) = \frac{C\omega^2\sigma_0^2}{1+C^2} - \tan^{-1}C.$$

- All averages can be performed analytically. The final result is found to be

$$\frac{\sigma^2}{\sigma_0^2} = \left(1 + \frac{C\beta_2 L}{2\sigma_0^2}\right)^2 + \left(\frac{\beta_2 L}{2\sigma_0^2}\right)^2 + \left(\frac{\beta_3 L(1+C^2)}{4\sqrt{2}\sigma_0^3}\right)^2.$$

Effects of Source Spectrum

- We have assumed that source spectrum is much narrower than the pulse spectrum.
- This condition is satisfied in practice for DFB lasers but not for light-emitting diodes.
- To account source spectral width, we must consider the coherence properties of the source.
- Input field $A(0,t) = A_0(t)a_p(t)$, where $a_p(t)$ represents pulse shape and fluctuations in $A_0(t)$ produce source spectrum.
- To account for source fluctuations,we replace $\langle t \rangle$ and $\langle t^2 \rangle$ with $\langle\langle t \rangle\rangle_s$ and $\langle\langle t^2 \rangle\rangle_s$, where outer brackets stand for an ensemble average over source fluctuations.

Effects of Source Spectrum

- $S(\omega)$ becomes a convolution of the pulse and source spectra

$$S(\omega) = \int_{-\infty}^{\infty} S_p(\omega - \omega_1) F(\omega_1)\, d\omega_1.$$

- $F(\omega)$ is the Fourier transform of $A_0(t)$ and satisfies

$$\langle F^*(\omega_1) F(\omega_2) \rangle_s = G(\omega_1) \delta(\omega_1 - \omega_2).$$

- Source spectrum $G(\omega)$ assumed to be Gaussian:

$$G(\omega) = \frac{1}{\sigma_\omega \sqrt{2\pi}} \exp\left(-\frac{\omega^2}{2\sigma_\omega^2}\right).$$

- σ_ω is the RMS spectral width of the source.

Effects of Source Spectrum

- For a chirped Gaussian pulse,all integrals can be calculated analytically as they involve only Gaussian functions.
- It is possible to obtain $\langle\langle t\rangle\rangle_s$ and $\langle\langle t^2\rangle\rangle_s$ in a closed form.
- RMS width of the pulse at the end of a fiber of length L:

$$\frac{\sigma_p^2}{\sigma_0^2} = \left(1+\frac{C\beta_2 L}{2\sigma_0^2}\right)^2 + (1+V_\omega^2)\left(\frac{\beta_2 L}{2\sigma_0^2}\right)^2 + \tfrac{1}{2}(1+C^2+V_\omega^2)^2\left(\frac{\beta_3 L}{4\sigma_0^3}\right)^2.$$

- $V_\omega = 2\sigma_\omega\sigma_0$ is a dimensionless parameter for a source with RMS width σ_ω.
- This equation provides an expression for dispersion-induced pulse broadening under general conditions.

Dispersion Limitations

Large Source Spectral Width: $V_\omega \gg 1$

- Assume input pulse to be unchirped ($C = 0$).
- Set $\beta_3 = 0$ when $\lambda \neq \lambda_{\text{ZD}}$:

$$\sigma^2 = \sigma_0^2 + (\beta_2 L \sigma_\omega)^2 \equiv \sigma_0^2 + (DL\sigma_\lambda)^2.$$

- 96% of pulse energy remains within the bit slot if $4\sigma < T_B = 1/B$.
- Using $4B\sigma \leq 1$, and $\sigma \gg \sigma_0$, $BL|D|\sigma_\lambda \leq \frac{1}{4}$.
- Set $\beta_2 = 0$ when $\lambda = \lambda_{\text{ZD}}$ to obtain

$$\sigma^2 = \sigma_0^2 + \tfrac{1}{2}(\beta_3 L \sigma_\omega^2)^2 \equiv \sigma_0^2 + \tfrac{1}{2}(SL\sigma_\lambda^2)^2.$$

- Dispersion limit: $BL|S|\sigma_\lambda^2 \leq 1/\sqrt{8}$.

Dispersion Limitations

Small Source Spectral Width: $V_\omega \ll 1$

- When $\beta_3 = 0$ and $C = 0$,

$$\sigma^2 = \sigma_0^2 + (\beta_2 L / 2\sigma_0)^2.$$

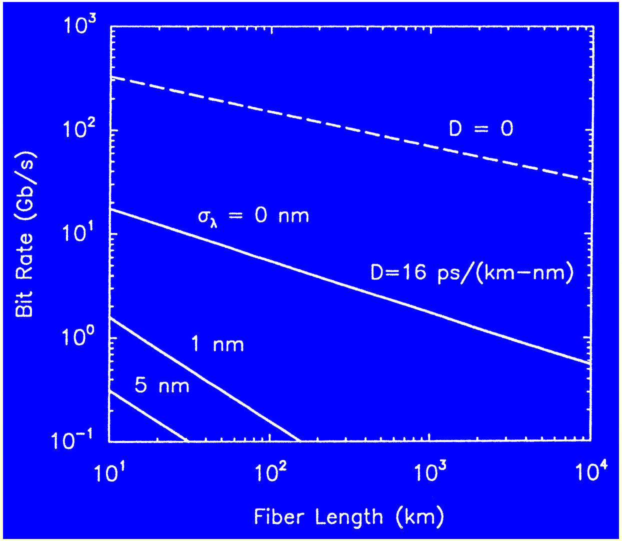
- One can minimize σ by adjusting input width σ_0.
- Minimum occurs for $\sigma_0 = (|\beta_2| L/2)^{1/2}$ and leads to $\sigma = (|\beta_2| L)^{1/2}$.
- Dispersion limit when $\beta_3 = 0$:

$$B\sqrt{|\beta_2| L} \leq \tfrac{1}{4}.$$

- Dispersion limit when $\beta_2 = 0$:

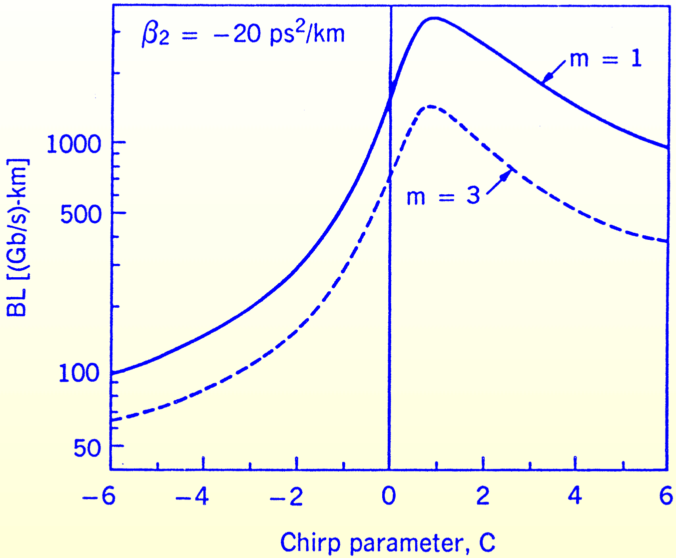
$$B(|\beta_3| L)^{1/3} \leq 0.324.$$

Dispersion Limitations (cont.)



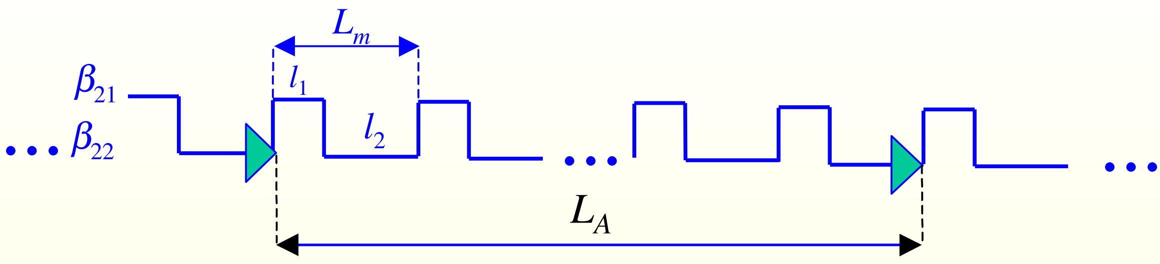
- Even a 1-nm spectral width limits $BL < 0.1$ (Gb/s)-km.
- DFB lasers essential for most lightwave systems.
- For $B > 2.5$ Gb/s, dispersion management required.

Effect of Frequency chirp



- Numerical simulations necessary for more realistic pulses.
- Super-Gaussian pulse: $A(0,T) = A_0 \exp\left[-\frac{1+iC}{2}\left(\frac{t}{T_0}\right)^{2m}\right]$.
- Chirp can affect system performance drastically.

Dispersion compensation



- Dispersion is a major limiting factor for long-haul systems.
- A simple solution exists to the dispersion problem.
- Basic idea: Compensate dispersion along fiber link in a periodic fashion using fibers with opposite dispersion characteristics.
- Alternate sections with normal and anomalous GVD are employed.
- Periodic arrangement of fibers is referred to as a dispersion map.

Condition for Dispersion Compensation

- GVD anomalous for standard fibers in the 1.55-μm region.
- Dispersion-compensating fibers (DCFs) with $\beta_2 > 0$ have been developed for dispersion compensation.
- Consider propagation of optical signal through one map period:

$$A(L_m,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{A}(0,\omega)\exp\left[\frac{i}{2}\omega^2(\beta_{21}l_1 + \beta_{22}l_2) - i\omega t\right] d\omega.$$

- If second fiber is chosen such that the phase term containing ω^2 vanishes, $A(L_m,t) \equiv A(0,t)$ (Map period $L_m = l_1 + l_2$).
- Condition for perfect dispersion compensation:

$$\beta_{21}l_1 + \beta_{22}l_2 = 0 \quad \text{or} \quad D_1l_1 + D_2l_2 = 0.$$

Polarization-Mode Dispersion

- State of polarization (SOP) of optical signal does not remain fixed in practical optical fibers.
- It changes randomly because of fluctuating birefringence.
- Geometric birefringence: small departures in cylindrical symmetry during manufacturing (fiber core slightly elliptical).
- Both the ellipticity and axes of the ellipse change randomly along the fiber on a length scale ~10 m.
- Second source of birefringence: anisotropic stress on the fiber core during manufacturing or cabling of the fiber.
- This type of birefringence can change with time because of environmental changes on a time scale of minutes or hours.

PMD Problem

- SOP of light totally unpredictable at any point inside the fiber.
- Changes in the SOP of light are not of concern because photodetectors respond to total power irrespective of SOP.
- A phenomenon known as polarization-mode dispersion (PMD) induces pulse broadening.
- Amount of pulse broadening can fluctuate with time.
- If the system is not designed with the worst-case scenario in mind, PMD- can move bits outside of their allocated time slots, resulting in system failure in an unpredictable manner.
- Problem becomes serious as the bit rate increases.

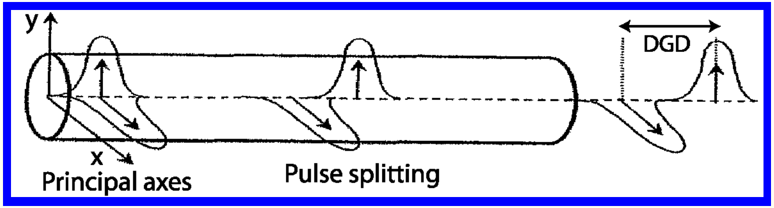
Fibers with Constant Birefringence

- Consider first polarization-maintaining fibers.
- Large birefringence intentionally induced to mask small fluctuations resulting from manufacturing and environmental changes.
- A single-mode fiber supports two orthogonally polarized modes.
- Two modes degenerate in all respects for perfect fibers.
- Birefringence breaks this degeneracy.
- Two modes propagate inside fiber with slightly different propagation constants ($\bar{n}$ slightly different).
- Index difference $\Delta n = \bar{n}_x - \bar{n}_y$ is a measure of birefringence.
- Two axes along which the modes are polarized are known as the principal axis.

Polarization-Maintaining Fibers

- When input pulse is polarized along a principal axis, its SOP does not change because only one mode is excited.
- Phase velocity $v_p = c/\bar{n}$ and group velocity $v_g = c/\bar{n}_g$ different for two principal axes.
- x axes chosen along principal axis with larger mode index.
- It is called the slow axis; y axis is the fast axis.
- When input pulse is not polarized along a principal axis, its energy is divided into two polarization modes.
- Both modes are equally excited when input pulse is polarized linearly at 45°.
- Two orthogonally polarized components of the pulse separate along the fiber because of their different group velocities.

Pulse Splitting



- Two components arrive at different times at the end of fiber.
- Single pulse splits into two pulses that are orthogonally polarized.
- Delay $\Delta\tau$ in the arrival of two components is given by

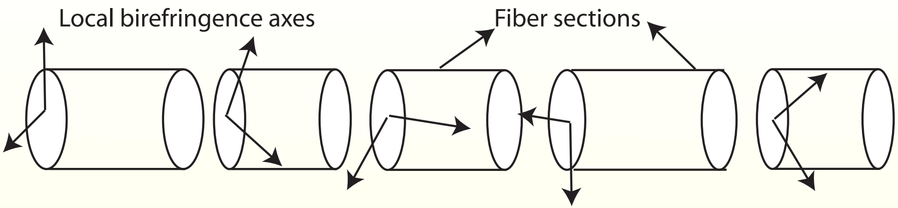
$$\Delta\tau = \left|\frac{L}{v_{gx}} - \frac{L}{v_{gy}}\right| = L|\beta_{1x} - \beta_{1y}| = L(\Delta\beta_1).$$

- $\Delta\beta_1 = \Delta\tau/L = v_{gx}^{-1} - v_{gy}^{-1}$.
- $\Delta\tau$ is called the differential group delay (DGD).
- For a PMF, $\Delta\beta_1 \sim 1$ ns/km.

Fibers with Random Birefringence

- Conventional fibers exhibit much smaller birefringence ($\Delta n \sim 10^{-7}$).
- Birefringence magnitude and directions of principal axes change randomly at a length scale (correlation length) $l_c \sim 10$m.
- SOP of light changes randomly along fiber during propagation.
- What affects the system is not a random SOP but pulse distortion induced by random changes in the birefringence.
- Two components of the pulse perform a random walk, each one advancing or retarding in a random fashion.
- Final separation $\Delta\tau$ becomes unpredictable, especially if birefringence fluctuates because of environmental changes.

Theoretical Model



- Birefringence remains constant in each section but changes randomly from section to section.
- Introduce the Ket vector $|\tilde{A}(z,\omega)\rangle = \begin{pmatrix} \tilde{A}_x(z,\omega) \\ \tilde{A}_y(z,\omega) \end{pmatrix}$.
- Propagation of each frequency component governed by a composite Jones matrix obtained by multiplying individual Jones matrices for each section:

$$|\tilde{A}(L,\omega)\rangle = T_N T_{N-1} \cdots T_2 T_1 |\tilde{A}(0,\omega)\rangle \equiv T_c(\omega)|\tilde{A}(0,\omega)\rangle.$$

Principal States of Polarization

- Two special SOPs, known as PSPs, exist for any fiber.
- When a pulse is polarized along a PSP, the SOP at the output of fiber is independent of frequency to first order.
- PSPs are analogous to the slow and fast axes associated with PMFs.
- They are in general elliptically polarized.
- An optical pulse polarized along a PSP does not split into two parts and maintains its shape.
- DGD $\Delta\tau$ is defined as the relative delay in the arrival time of pulses polarized along the two PSPs.
- PSPs and $\Delta\tau$ change with L in a a random fashion.

PMD-Induced Pulse Broadening

- Optical pulses are rarely polarized along one of PSPs.
- Each pulse splits into two parts that are delayed by a random amount $\Delta\tau$.
- PMD-induced pulse broadening is characterized by RMS value of $\Delta\tau$ obtained after averaging over random birefringence.
- Several approaches have been used to calculate this average.
- The second moment of $\Delta\tau$ is given by

$$\langle(\Delta\tau)^2\rangle \equiv \Delta\tau_{\text{RMS}}^2 = 2(\Delta\beta_1)^2 l_c^2[\exp(-z/l_c) + z/l_c - 1].$$

- l_c is the length over which two polarization components remain correlated.

PMD Parameter

- RMS Value of DGD depends on distance z as

$$\Delta\tau^2_{\text{RMS}}(z) = 2(\Delta\beta_1)^2 l_c^2[\exp(-z/l_c) + z/l_c - 1].$$

- For short distances such that $z \ll l_c$, $\Delta\tau_{\text{RMS}} = (\Delta\beta_1)z$.
- This is expected since birefringence remains constant.
- For $z \gg 1$ km, take the limit $z \gg l_c$:

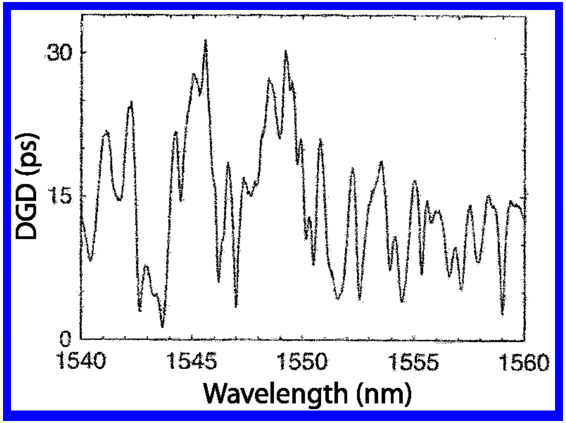
$$\Delta\tau_{\text{RMS}} \approx (\Delta\beta_1)\sqrt{2l_c z} \equiv D_p\sqrt{z}.$$

- Square-root dependence on length expected for a random walk.
- D_p is known as the PMD parameter.

Impact of PMD Parameter

- Measured values of D_p vary from fiber to fiber in the range $D_p = $ 0.01–10 ps/$\sqrt{\text{km}}$.
- Fibers installed during the 1980s had a relatively large PMD.
- Modern fibers are designed to have low PMD; typically $D_p < 0.1$ ps/$\sqrt{\text{km}}$.
- PMD-induced pulse broadening relatively small compared with GVD effects.
- For example, $\Delta\tau_{\text{RMS}} = 1$ ps for a fiber length of 100 km, if we use $D_p = 0.1$ ps/$\sqrt{\text{km}}$.
- PMD becomes a limiting factor for systems designed to operate over long distances at high bit rates.

DGD Fluctuations



- Experimental data over a 20-nm-wide range for a fiber with mean DGD of 14.7 ps.
- DGD fluctuates with the wavelength of light.
- Measured values of DGD vary randomly from 2 ps to more than 30 ps depending on wavelength of light.

Polarization-Dependent Losses

- Losses of a fiber link often depend on SOP of the signal propagating through it.
- Silica fibers themselves exhibit little PDL.
- Optical signal passes through a variety of optical components (isolators, modulators, amplifiers, filters, couplers, etc.)
- Most components exhibit loss (or gain) whose magnitude depends on the SOP of the signal.
- PDL is relatively small for each component (∼0.1 dB).
- Cumulative effect of all components produces an output signal whose power may fluctuate by a factor of 10 or more depending on its input SOP.

Chapter 4: Nonlinear Impairments

- Inclusion of nonlinear effects essential for long-haul systems employing a chain of cascaded optical amplifiers.
- Noise added by the amplifier chain degrades the SNR and requires high launched powers.
- Nonlinear effects accumulate over multiple amplifiers and distort the bit stream.
- Five Major Nonlinear Effects are possible in optical fibers:
  - ⋆ Stimulated Raman Scattering (SRS)
  - ⋆ Stimulated Brillouin Scattering (SBS)
  - ⋆ Self-Phase Modulation (SPM)
  - ⋆ Cross-Phase Modulation (XPM)
  - ⋆ Four-Wave Mixing (FWM)

Self-Phase Modulation

- Pulse propagation inside an optical fiber is governed by

$$\frac{\partial A}{\partial z} + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} = i\gamma|A|^2A + \frac{1}{2}[g_0(z) - \alpha]A.$$

- Eliminate gain–loss terms using $A(z,t) = \sqrt{P_0 p(z)}\,U(z,t)$.
- $p(z)$ takes into account changes in average power of signal along the fiber link; it is defined such that $p(nL_A) = 1$.
- $U(z,t)$ satisfies the NLS equation

$$\frac{\partial U}{\partial z} + \frac{i\beta_2}{2}\frac{\partial^2 U}{\partial t^2} = i\gamma P_0 p(z)|U|^2U.$$

- Last term leads to Self-Phase Modulation (SPM).

Nonlinear Phase Shift

- In the limit $\beta_2 = 0$, $\frac{\partial U}{\partial z} = \frac{ip(z)}{L_{\rm NL}}|U|^2 U$.
- Nonlinear length is defined as $L_{\rm NL} = 1/(\gamma P_0)$.
- It provides a length scale over which nonlinear effects become relevant.
- As an example, if $\gamma = 2$ W^{-1}/km, $L_{\rm NL} = 100$ km for $P_0 = 5$ mW.
- Using $U = V \exp(i\phi_{\rm NL})$, we obtain

$$\frac{\partial V}{\partial z} = 0, \qquad \frac{\partial \phi_{\rm NL}}{\partial z} = \frac{p(z)}{L_{\rm NL}} V^2.$$

- General solution: $U(L,t) = U(0,t)\exp[i\phi_{\rm NL}(L,t)]$.
- Nonlinear Phase Shift: $\phi_{\rm NL}(L,t) = |U(0,t)|^2 (L_{\rm eff}/L_{\rm NL})$.
- Effective fiber length $L_{\rm eff} = \int_0^L p(z)\,dz = N_A \int_0^{L_A} p(z)\,dz$.

Self-Phase Modulation

- Nonlinear term in the NLS equation leads to an intensity-dependent phase shift.
- This phenomenon is referred to as SPM because the signal modulates its own phase.
- It was first observed in a 1978 experiment.
- Nonlinear phase shift $\phi_{\rm NL} = |U(0,t)|^2 (L_{\rm eff}/L_{\rm NL})$.
- Maximum phase shift $\phi_{\max} = L_{\rm eff}/L_{\rm NL} = \gamma P_0 L_{\rm eff}$.
- $L_{\rm eff}$ is smaller than L because of fiber losses.
- In the case of lumped amplification, $p(z) = \exp(-\alpha z)$.
- Effective link length $L_{\rm eff} = L[1-\exp(-\alpha L_A)]/(\alpha L_A) \approx N_A/\alpha$.
- In the absence of fiber losses, $L_{\rm eff} = L$.

SPM-Induced Chirp

- Consider a single 1 bit within an RZ bit stream.
- A temporally varying phase implies that carrier frequency differs across the pulse from its central value ω_0.
- The frequency shift $\delta\omega$ is itself time-dependent:

$$\delta\omega(t) = -\frac{\partial \phi_{\rm NL}}{\partial t} = -\left(\frac{L_{\rm eff}}{L_{\rm NL}}\right)\frac{\partial}{\partial t}|U(0,t)|^2.$$

- Minus sign is due to the choice $\exp(-i\omega_0 t)$.
- $\delta\omega(t)$ is referred to as the frequency chirp.
- New frequency components are generated continuously as signal propagates down the fiber.
- New frequency components broaden spectrum of the bit stream.

SPM-Induced Chirp

- For a random bit sequence $U(0,t) = \sum b_n U_p(t - nT_b)$.
- SPM-induced phase shift can be written as

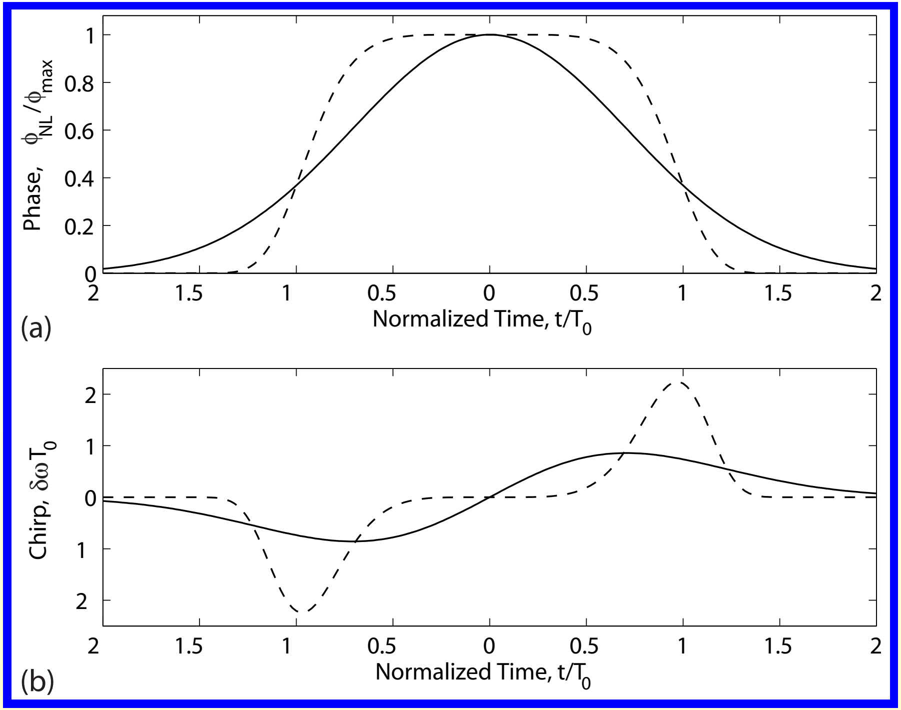
$$\phi_{\rm NL}(L,t) \approx (L_{\rm eff}/L_{\rm NL}) \sum_k b_k^2 |U_p(t - kT_b)|^2.$$

- Nonlinear phase shift occurs for only 1 bits.
- The form of $\phi_{\rm NL}$ mimics the bit pattern of the launched signal.
- Magnitude of SPM-induced chirp depends on pulse shape.
- In the case of a super-Gaussian pulse

$$\delta\omega(t) = \frac{2m}{T_0}\frac{L_{\rm eff}}{L_{\rm NL}}\left(\frac{t}{T_0}\right)^{2m-1} \exp\left[-\left(\frac{t}{T_0}\right)^{2m}\right].$$

- Integer $m = 1$ for a Gaussian pulse.

SPM-Induced Chirp



- ϕ_{NL} and $\delta\omega$ across the pulse at a distance $L_{eff} = L_{NL}$ for Gaussian ($m = 1$) and super-Gaussian ($m = 3$) pulses (dashed curves).

Spectral Broadening and Narrowing

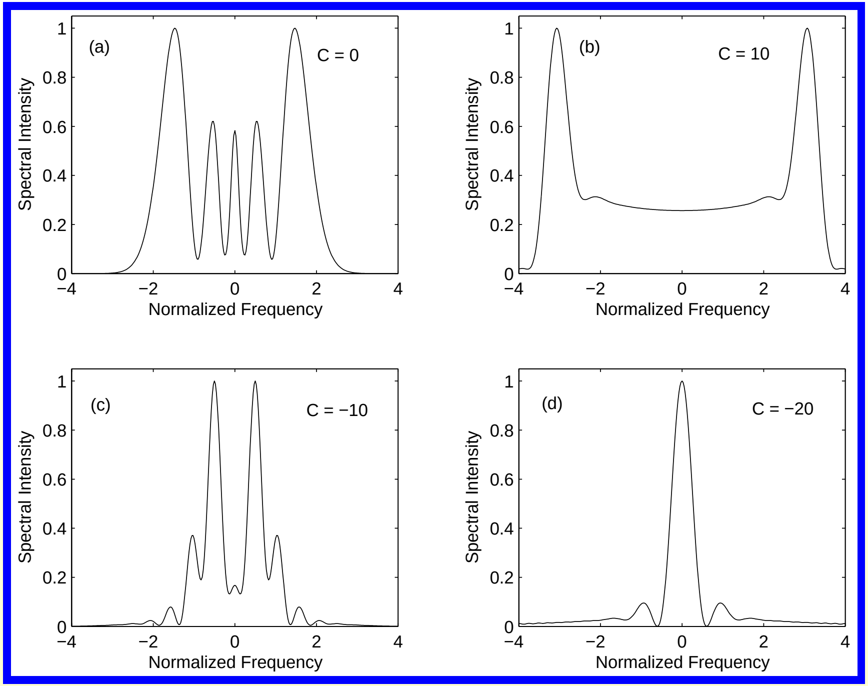
- Spectrum of a bit stream changes as it travels down the link.
- SPM-induced spectral broadening can be estimated from $\delta\omega(t)$.
- Maximum value $\delta\omega_{\max} = mf(m)\phi_{\max}/T_0$:

$$f(m) = 2\left(1-\frac{1}{2m}\right)^{1-1/2m}\exp\left[-\left(1-\frac{1}{2m}\right)\right].$$

- $f(m) = 0.86$ for $m = 1$ and tends toward 0.74 for $m > 1$.
- Using $\Delta\omega_0 = T_0^{-1}$ with $m = 1$, $\delta\omega_{\max} = 0.86\Delta\omega_0\phi_{\max}$.
- Spectral shape of at a distance L is obtained from

$$S(\omega) = \left|\int_{-\infty}^{\infty} U(0,t)\exp[i\phi_{\rm NL}(L,t) + i(\omega-\omega_0)t]\,dt\right|^2.$$

Pulse Spectra and Input Chirp



- Gaussian-pulse spectra for 4 values of C when $\phi_{\max} = 4.5\pi$.
- Spectrum broadens for $C < 0$ but becomes narrower for $C < 0$.

Control of SPM

- Sign of the chirp parameter C plays a critical role.
- A negatively chirped pulse undergoes spectral narrowing.
- This behavior can be understood by noting that SPM-induced chirp is partially cancelled by when C is negative.
- If we use $\phi_{\rm NL}(t) \approx \phi_{\max}(1 - t^2/T_0^2)$ for Gaussian pulses, SPM-induced chirp is nearly cancelled for $C = -2\phi_{\max}$.
- SPM-induced spectral broadening should be controlled for any system.
- As a rough design guideline, SPM effects become important only when $\phi_{\max} > 1$. This condition satisfied if $P_0 < \alpha/(\gamma N_A)$.
- For typical values of α and γ, peak power is limited to below 1 mW for a fiber links containing 30 amplifiers.

Effect of Dispersion

- Dispersive and nonlinear effects act on bit stream simultaneously.
- One must solve the NLS equation:

$$\frac{\partial U}{\partial z} + \frac{i\beta_2}{2}\frac{\partial^2 U}{\partial t^2} = i\gamma P_0 p(z)|U|^2 U.$$

- When $p = 1$ and $\beta_2 < 0$, the NLS equation has solutions in the form of solitons.
- Solitons are pulses that maintain their shape and width in spite of dispersion.
- Another special case is that of "rect" pulses propagating in a fiber with $\beta_2 > 0$ (normal GVD).
- This problem is identical to the hydrodynamic problem of "breaking a dam."

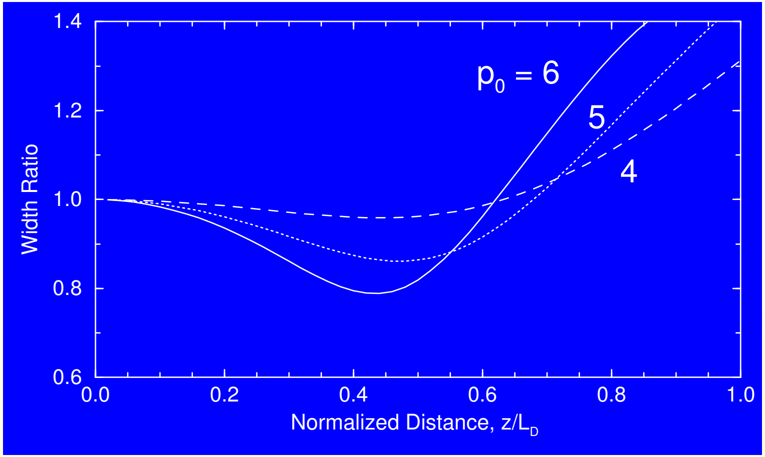
Pulse Broadening Revisited

- SPM-induced chirp affects broadening of optical pulses.
- Broadening factor can be estimated without a complete solution.
- A perturbative approach yields:

$$\sigma_p^2(z) = \sigma_L^2(z) + \gamma P_0 f_s \int_0^z \beta_2(z_1) \left[\int_0^{z_1} p(z_2)\, dz_2 \right] dz_1.$$

- σ_L^2 is the RMS width expected in the linear case ($\gamma = 0$).
- Shape of input pulse enters through $f_s = \frac{\int_{-\infty}^{\infty} |U(0,t)|^4\, dt}{\int_{-\infty}^{\infty} |U(0,t)|^2\, dt}$.
- For a Gaussian pulse, $f_s = 1/\sqrt{2} \approx 0.7$. For a square pulse, $f_s = 1$.
- For $p(z) = 1$ and constant β_2, $\sigma_p^2(z) = \sigma_L^2(z) + \frac{1}{2}\gamma P_0 f_s \beta_2 z^2$.

Pulse Spectra and Input Chirp



- Width ratio σ_p/σ_0 as a function of propagation distance for a super-Gaussian pulse $(m = 2,\ p_0 = \gamma P_0 L_D)$.
- SPM enhances pulse broadening when $\beta_2 > 0$ but leads to pulse compression in the case of anomalous GVD.
- This behavior can be understood by noting that SPM-induced chirp is positive $(C > 0)$.

Modulation Instability

- Modulation instability is an instability of the CW solution of the NLS equation in the anomalous-GVD regime.
- CW solution has the form $U(z) = \exp(i\phi_{\text{NL}})$.
- Perturb the CW solution such that $U = (1+a)\exp(i\phi_{\text{NL}})$.
- Linearizing in a we obtain

$$i\frac{\partial a}{\partial z} = \frac{\beta_2}{2}\frac{\partial^2 a}{\partial t^2} - \gamma P_0(a+a^*).$$

- This linear equation has solution in the form

$$a(z,t) = a_1 \exp[i(Kz-\Omega t)] + a_2 \exp[-i(Kz-\Omega t)],$$

- Solution exists only if $K = \frac{1}{2}|\beta_2\Omega|[\Omega^2 + \text{sgn}(\beta_2)\Omega_c^2]^{1/2}$, where

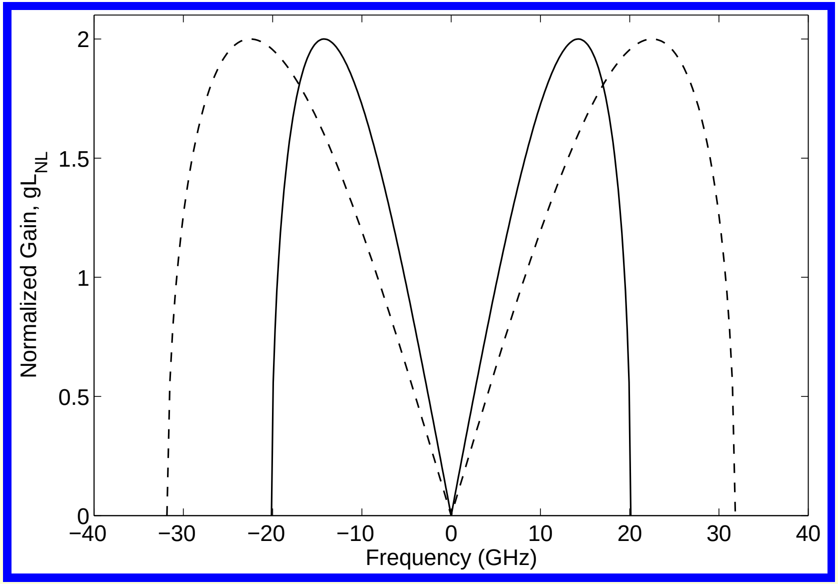
$$\Omega_c^2 = \frac{4\gamma P_0}{|\beta_2|} = \frac{4}{|\beta_2|L_{\text{NL}}}.$$

Gain Spectrum

- Dispersion relation: $K = \frac{1}{2}|\beta_2\Omega|[\Omega^2 + \text{sgn}(\beta_2)\Omega_c^2]^{1/2}$.
- CW solution is unstable when K becomes complex because any perturbation then grows exponentially.
- Stability depends on whether light experiences normal or anomalous GVD inside the fiber.
- In the case of normal GVD $(\beta_2 > 0)$, K is real for all Ω, and steady state is stable.
- When $\beta_2 < 0$, K becomes imaginary for $|\Omega| < \Omega_c$.
- Instability transforms a CW beam into a pulse train.
- Gain $g(\Omega) = 2\,\text{Im}(K)$ exists only for $|\Omega| < \Omega_c$:

$$g(\Omega) = |\beta_2\Omega|(\Omega_c^2 - \Omega^2)^{1/2}.$$

Gain Spectrum



- $L_{\rm NL} = 20$ km (dashed curve) or 50 km; $\beta_2 = -5$ ps^2/km.
- Gain peaks at frequencies $\Omega_{\max} = \pm\frac{\Omega_c}{\sqrt{2}} = \pm\sqrt{2\gamma P_0/|\beta_2|}$.
- Peak value $g_{\max} \equiv g(\Omega_{\max}) = \frac{1}{2}|\beta_2|\Omega_c^2 = 2\gamma P_0$.

Impact of Modulation Instability

- Modulation instability affects the performance of periodically amplified lightwave systems.
- It can be seeded by broadband noise added by amplifiers.
- Growth of this noise degrades the SNR at the receiver end.
- In the case of anomalous GVD, spectral components of noise falling within the gain bandwidth are amplified.
- SPM-induced reduction in signal SNR has been observed in several experiments.
- Use of optical filters after each amplifier helps in practice.

Cross-Phase Modulation

- Nonlinear refractive index seen by one wave depends on the intensity of other copropagating channels.
- Nonlinear index for two channels:

$$\Delta n_{\mathrm{NL}} = n_2(|A_1|^2 + 2|A_2|^2).$$

- Total nonlinear phase shift for multiple channels:

$$\phi_j^{\mathrm{NL}} = \gamma L_{\mathrm{eff}} \left(P_j + 2 \sum_{m \neq j} P_m \right).$$

- XPM induces a nonlinear coupling among channels.
- XPM is a major source of crosstalk in WDM systems.

Coupled NLS Equations

- Consider a two-channel lightwave system with total field

$$A(z,t) = A_1(z,t)\exp(-i\Omega_1 t) + A_2(z,t)\exp(-i\Omega_2 t).$$

- Substituting it in the NLS equation, we obtain:

$$\frac{\partial A_1}{\partial z} + \Omega_1\beta_2\frac{\partial A_1}{\partial t} + \frac{i\beta_2}{2}\frac{\partial^2 A_1}{\partial t^2} = i\gamma(|A_1|^2 + 2|A_2|^2)A_1 + \frac{i}{2}\beta_2\Omega_1^2 A_1$$

$$\frac{\partial A_2}{\partial z} + \Omega_2\beta_2\frac{\partial A_2}{\partial t} + \frac{i\beta_2}{2}\frac{\partial^2 A_2}{\partial t^2} = i\gamma(|A_2|^2 + 2|A_1|^2)A_2 + \frac{i}{2}\beta_2\Omega_2^2 A_2.$$

- Single nonlinear term $|A|^2 A$ gives rise to two nonlinear terms.
- Second term is due to XPM and produces a nonlinear phase shift that depends on the power of the other channel.

XPM-Induced Phase Shift

- For an M-channel WDM system, we obtain M equations of the form

$$\frac{\partial A_j}{\partial z} + \Omega_j \beta_2 \frac{\partial A_j}{\partial t} + \frac{i\beta_2}{2}\frac{\partial^2 A_j}{\partial t^2} = i\gamma\Big(|A_j|^2 + 2\sum_{m\neq j}|A_m|^2\Big)A_j + \frac{i}{2}\beta_2\Omega_j^2 A_j.$$

- These equations can be solved analytically in the CW case or when the dispersive effects are ignored.
- Setting $\beta_2 = 0$ and integrating over z, we obtain

$$A_j(L) = \sqrt{P_j}\exp(i\phi_j), \qquad \phi_j = \gamma L_{\text{eff}}\Big(P_j + 2\sum_{m\neq j} P_m\Big).$$

- XPM phase shift depends on powers of all other channels.

Limitation on Channel powers

- Assume input power is the same for all channels.
- Maximum value of phase shift occurs when 1 bits in all channels overlap simultaneously.
- $\phi_{\max} = N_A(\gamma/\alpha)(2M-1)P_{\rm ch}$, where $L_{\rm eff} = N_A/\alpha$ was used.
- XPM-induced phase shift increases linearly both with M and N_A.
- It can become quite large for long-haul WDM systems.
- If we use $\phi_{\max} < 1$ and $N_A = 1$, channel power is restricted to $P_{\rm ch} < \alpha/[\gamma(2M-1)]$.
- For typical values of α and γ, $P_{\rm ch} < 10$ mW even for five channels.
- Allowed power level reduces to below 1 mW for >50 channels.

Effects of Group-Velocity Mismatch

- Preceding analysis overestimates the XPM-induced phase shift.
- Pulses belonging to different channels travel at different speeds.
- XPM can occur only when pulses overlap in the time domain.
- XPM-induced phase shift induced is reduced considerably by the walk-off effects.
- Consider a pump-probe configuration in which one of the channels is in the form of a weak CW field.
- If we neglect dispersion, XPM coupling is governed by

$$\frac{\partial A_1}{\partial z} + \delta \frac{\partial A_1}{\partial t} = i\gamma |A_1|^2 A_1 + \frac{i}{2}\beta_2 \Omega_1^2 A_1 - \frac{\alpha}{2} A_1.$$

$$\frac{\partial A_2}{\partial z} = 2i\gamma |A_1|^2 A_2 - \frac{\alpha}{2} A_2, \qquad \delta = \Omega_1 \beta_2.$$

Pump-Probe Configuration

- Introducing $A_1 = \sqrt{P_1}\exp(i\phi_1)$, pump power $P_1(z,t)$ satisfies $\frac{\partial P_1}{\partial z} + \delta\frac{\partial P_1}{\partial t} + \alpha A_1 = 0$.
- Its solution is $P_1(z,t) = P_{\rm in}(t-\delta z)e^{-\alpha z}$.
- Probe equation can also be solved to obtain $A_2(z) = A_2(0)\exp(-\alpha L/2 + i\phi_{\rm XPM})$.
- XPM-induced phase shift is given by

$$\phi_{\rm XPM}(t) = 2\gamma\int_0^L P_{\rm in}(t-\delta z)e^{-\alpha z}\,dz.$$

- For a CW pump we recover the result obtained earlier.
- For a time-dependent pump, phase shift is affected considerably by group-velocity mismatch through δ.

Effects of Group-Velocity Mismatch

- Consider a pump modulated sinusoidally at ω_m as

$$P_{\rm in}(t) = P_0 + p_m \cos(\omega_m t).$$

- Writing probe's phase shift in the form $\phi_{\rm XPM} = \phi_0 + \phi_m \cos(\omega_m t + \psi)$, we find $\phi_0 = 2\gamma P_0 L_{\rm eff}$ and

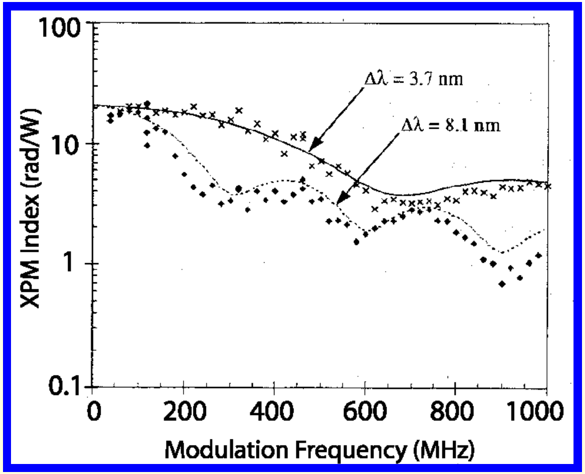
$$\phi_m(\omega_m) = 2\gamma p_m L_{\rm eff}\sqrt{\eta_{\rm XPM}}.$$

- $\eta_{\rm XPM}$ is a measure of the XPM efficiency:

$$\eta_{\rm XPM}(\omega_m) = \frac{\alpha^2}{\alpha^2 + \omega_m^2\delta^2}\left[1 + \frac{4\sin^2(\omega_m \delta L/2)e^{-\alpha L}}{(1-e^{-\alpha L})^2}\right].$$

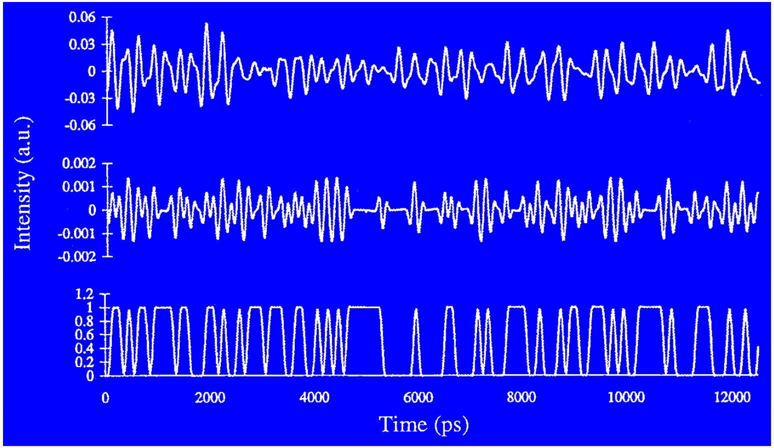
- Phase shift $\phi_m(\omega_m)$ depends on ω_m but also on channel spacing Ω_1 through δ.

Effects of Group-Velocity Mismatch



- XPM index, defined as ϕ_m/p_m, plotted as a function of ω_m for two different channel spacings.
- Experiment used 25-km-long single-mode fiber with 16.4 ps/(km-nm) dispersion and 0.21 dB/km losses.
- Experimental results agree well with theoretical predictions.

XPM-Induced Power Fluctuations



- XPM-induced phase shift should not affect a lightwave system because receivers respond to only channel powers.
- Dispersion converts pattern-dependent phase shifts into power fluctuations, resulting in a lower SNR.
- Power fluctuations at 130 (middle) and 320 km (top).
- Bit stream in the pump channel is shown at bottom.

XPM-Induced Timing Jitter

- Combination of GVD and XPM also leads to timing jitter.
- Frequency chirp induced by XPM depends on dP/dt.
- This derivative has opposite signs at leading and trailing edges.
- As a result, pulse spectrum first shifts toward red and then toward blue.
- In a lossless fiber, collisions are perfectly symmetric, resulting in no net spectral shift at the end of the collision.
- Amplifiers make collisions asymmetric, resulting in a net frequency shift that depends on channel spacing.
- Such frequency shifts lead to timing jitter (the speed of a channel depends on its frequency because of GVD).

Control of XPM Interaction

- Dominant contribution to XPM for any channel comes from two channels that are its nearest neighbors.
- XPM interaction can be reduced by increasing channel spacing.
- A larger channel spacing increases the group velocity mismatch.
- As a result, pulses cross each other so fast that they overlap for a relatively short duration.
- This scheme is effective but it reduces spectral efficiency.
- XPM effects can also be reduced by lowering channel powers.
- This approach not practical because a reduction in channel power also lowers the SNR at the receiver.

Control of XPM Interaction

- A simple scheme controls the state of polarization (SOP) of neighboring channels.
- Channels are launched such that any two neighboring channels are orthogonally polarized.
- In practice, even- and odd-numbered channels are grouped together and their SOPs are made orthogonal.
- This scheme is referred to as polarization channel interleaving.
- XPM interaction between two orthogonally polarized is reduced significantly.
- Mathematically, the factor of 2 in the XPM-induced phase shift is replaced with 2/3.

Four-Wave Mixing (FWM)

- FWM is a process in which two photons of energies $\hbar\omega_1$ and $\hbar\omega_2$ are converted into two new photons of energies $\hbar\omega_3$ and $\hbar\omega_4$.
- Energy conservation: $\omega_1 + \omega_2 = \omega_3 + \omega_4$.
- Degenerate FWM: $2\omega_1 = \omega_3 + \omega_4$.
- Momentum conservation or phase matching is required.
- FWM efficiency governed by phase mismatch:

$$\Delta = \beta(\omega_3) + \beta(\omega_4) - \beta(\omega_1) - \beta(\omega_2).$$

- Propagation constant $\beta(\omega) = \bar{n}(\omega)\omega/c$ for a channel at ω.
- FWM becomes important for WDM systems designed with low-dispersion or dispersion-flattened fibers.

FWM-Induced Degradation

- FWM can generate a new wave at frequency $\omega_{\rm FWM} = \omega_i + \omega_j - \omega_k$ for any three channels at ω_i, ω_j, and ω_k.
- For an M-channel system, i, j, and k vary from 1 to M, resulting in a large combination of new frequencies.
- When channels are not equally spaced, most FWM components fall in between the channels and act as background noise.
- For equally spaced channels, new frequencies coincide with existing channel frequencies and interfere coherently with the signals in those channels.
- This interference depends on bit pattern and leads to considerable fluctuations in the detected signal at the receiver.
- System performance is degraded severely in this case.

FWM Equations

- Total optical field: $A(z,t) = \sum_{m=1}^{M} A_m(z,t)\exp(-i\Omega_m t)$.
- NLS equation for a pecific channel takes the form

$$\frac{\partial A_m}{\partial z} + \Omega_j \beta_2 \frac{\partial A_m}{\partial t} + \frac{i\beta_2}{2}\frac{\partial^2 A_m}{\partial t^2} = \frac{i}{2}\beta_2 \Omega_m^2 A_m - \frac{\alpha}{2} A_m$$

$$+ i\gamma\left(|A_m|^2 + 2\sum_{j\neq m}^{M} |A_j|^2\right) A_m + i\gamma \sum_i \sum_j \sum_k A_i A_j A_k^*.$$

- Triple sum restricted to only frequency combinations that satisfy $\omega_m = \omega_i + \omega_j - \omega_k$.
- Consider a single FWM term in the triple sum, focus on the quasi-CW case, and neglect phase shifts induced by SPM and XPM.
- Eliminate the remaining β_2 term through the transformation $A_m = B_m \exp(i\beta_2 \Omega_m^2 z/2 - \alpha z/2)$.

FWM Efficiency

- B_m satisfies the simple equation

$$\frac{dB_m}{dz} = i\gamma B_i B_j B_k^* \exp(-\alpha z - i\Delta k z), \quad \Delta k = \beta_2(\Omega_m^2 + \Omega_k^2 - \Omega_i^2 - \Omega_j^2).$$

- Power transferred to FWM component: $P_m = \eta_{\text{FWM}}(\gamma L)^2 P_i P_j P_k e^{-\alpha L}$, where $P_j = |A_j(0)|^2$ is the channel power.

- FWM efficiency is defined as

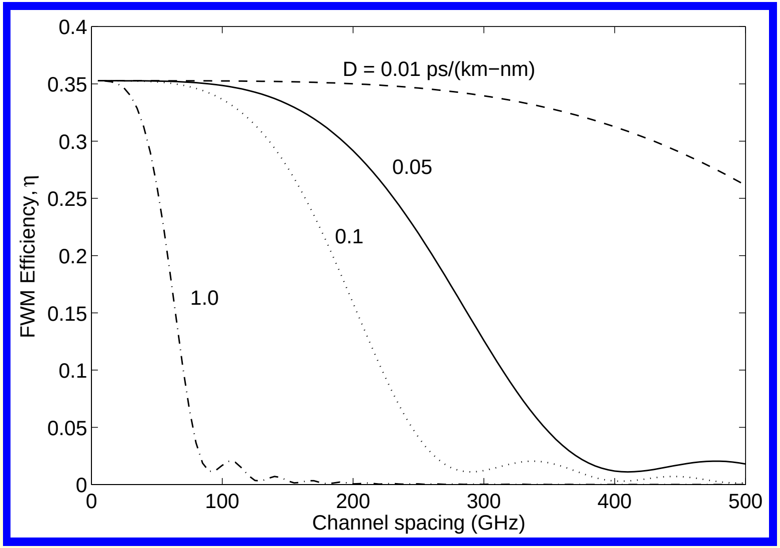
$$\eta_{\text{FWM}} = \left| \frac{1 - \exp[-(\alpha + i\Delta k)L]}{(\alpha + i\Delta k)L} \right|^2.$$

- η_{FWM} depends on channel spacing through phase mismatch Δk.

- Using $\Omega_m = \Omega_i + \Omega_j - \Omega_k$, this mismatch can be written as

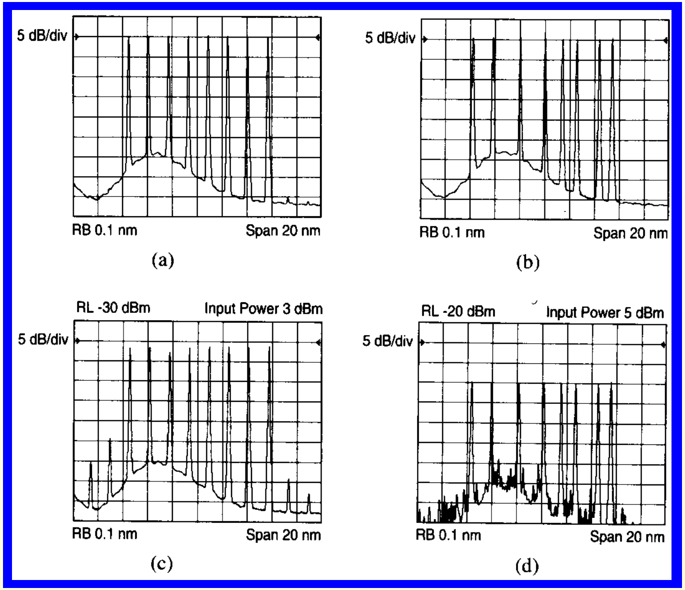
$$\Delta k = \beta_2(\Omega_i - \Omega_k)(\Omega_j - \Omega_k) \equiv \beta_2(\omega_i - \omega_k)(\omega_j - \omega_k).$$

FWM Efficiency



- Figure shows how $\eta_{\rm FWM}$ varies with $\Delta\nu_{\rm ch}$ for several values of D, using $\alpha = 0.2$ dB/km.
- FWM efficiency is relatively large for low-dispersion fibers.
- In contrast, $\eta_{\rm FWM} \approx 0$ for $\Delta\nu_{\rm ch} > 50$ GHz if $D > 2$ ps/(km-nm).

Channel Spectra



- Input (a) and output (c) optical spectra for eight equally spaced channels launched with 2-mW powers (link length 137 km).
- Input (b) and output (d) optical spectra in the case of unequal channel spacings.

Control of FWM

- Design WDM systems with unequal channel spacings.
- This scheme is not practical since many WDM components require equally spaced channels.
- Such a scheme is also spectrally inefficient.
- A practical solution offered by dispersion-management technique.
- Fibers with normal and anomalous GVD combined to form a periodic dispersion map.
- GVD is locally high in all fiber sections but its average value remains close to zero.
- By 1996, the use of dispersion management became common.
- All modern WDM systems make use of dispersion management.

FWM: Good or Bad?

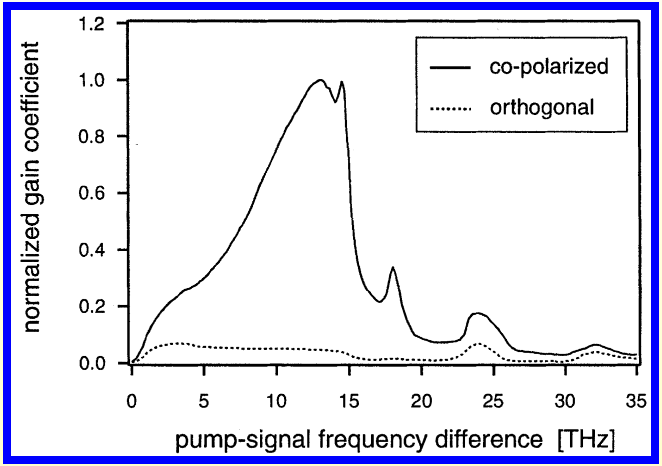
- FWM leads to interchannel crosstalk in WDM systems.
- It can be avoided through dispersion management.

On the other hand . . .
FWM can be used beneficially for

- Parametric amplification
- Optical phase conjugation
- Demultiplexing of OTDM channels
- Wavelength conversion of WDM channels
- Supercontinuum generation

Stimulated Raman Scattering (SRS)

- Scattering of light from vibrating molecules.
- Scattered light shifted in frequency.
- Raman gain spectrum extends over 40 THz.



- Raman shift at Gain peak: $\Omega_R = \omega_p - \omega_s \approx 13$ THz.

SRS Equations

- SRS governed by two coupled nonlinear equations:

$$\frac{dI_s}{dz} = g_R(\Omega) I_p I_s - \alpha_s I_s.$$

$$\frac{dI_p}{dz} = -\frac{\omega_p}{\omega_s} g_R(\Omega) I_p I_s - \alpha_p I_p$$

- Assume pump is so intense that its depletion can be ignored.
- Using $I_p(z) = I_0 \exp(-\alpha_p z)$, I_s satisfies

$$dI_s/dz = g_R I_0 \exp(-\alpha_p z) I_s - \alpha_s I_s.$$

- For a fiber of length L the solution is

$$I_s(L) = I_s(0) \exp(g_R I_0 L_{\text{eff}} - \alpha_s L).$$

- Effective fiber length $L_{\text{eff}} = (1 - e^{-\alpha L})/\alpha$.

Spontaneous Raman Scattering

- SRS builds up from spontaneous Raman scattering occurring all along the fiber.
- Equivalent to injecting one photon per mode at the input of fiber.
- Stokes power results from amplification of this photon over the entire bandwidth of Raman gain:

$$P_s(L) = \int_{-\infty}^{\infty} \hbar\omega \exp[g_R(\omega_p - \omega)I_0 L_{\text{eff}} - \alpha_s L]\, d\omega.$$

- Using the method of steepest descent we obtain

$$P_s(L) = P_{s0}^{\text{eff}} \exp[g_R(\Omega_R)I_0 L_{\text{eff}} - \alpha_s L],$$

- Effective input power at $z = 0$ is given by

$$P_{s0}^{\text{eff}} = \hbar\omega_s \left(\frac{2\pi}{I_0 L_{\text{eff}}}\right)^{1/2} \left(\frac{\partial^2 g_R}{\partial \omega^2}\right)^{-1/2}_{\omega=\omega_s}.$$

Raman Threshold

- Raman threshold: input pump power at which Stokes power equals pump power at the fiber output: $P_s(L) = P_p(L) \equiv P_0 \exp(-\alpha_p L)$.
- Assuming $\alpha_s \approx \alpha_p$, threshold condition becomes

$$P_{s0}^{\text{eff}} \exp(g_R P_0 L_{\text{eff}} / A_{\text{eff}}) = P_0.$$

- Assuming a Lorentzian shape for Raman gain spectrum, threshold power can be estimated from

$$\frac{g_R P_{th} L_{\text{eff}}}{A_{\text{eff}}} \approx 16.$$

- $L_{\text{eff}} \approx 1/\alpha$ for long fiber lengths.
- Using $g_R \approx 6 \times 10^{-14}$ m/W, P_{th} is about 500 mW near 1.55 μm.
- SRS is not of much concern for single-channel systems.

Raman Threshold

- Situation is quite different for WDM systems.
- Transmission fiber acts as a distributed Raman amplifier.
- Each channel amplified by all channels with a shorter wavelength as long as their wavelength difference is within Raman-gain bandwidth.
- Shortest-wavelength channel is depleted most as it can pump all other channels simultaneously.
- Such an energy transfer is detrimental because it depends on bit patterns of channels.
- It occurs only when 1 bits are present in both channels simultaneously and leads to power fluctuations.

Brillouin Scattering

- Scattering of light from self-induced acoustic waves.
- Energy and momentum conservation laws require $\Omega_B = \omega_p - \omega_s$ and $\mathbf{k}_A = \mathbf{k}_p - \mathbf{k}_s$.
- Brillouin shift: $\Omega_B = |k_A| v_A = 2 v_A |k_p| \sin(\theta/2)$.
- Only possibility $\theta = \pi$ for single-mode fibers (backward propagating Stokes wave).
- Using $k_p = 2\pi\bar{n}/\lambda_p$, $\nu_B = \Omega_B/2\pi = 2\bar{n} v_A/\lambda_p$.
- With $v_A = 5.96$ km/s and $\bar{n} = 1.45$, $\nu_B \approx 11$ GHz near 1.55 μm.
- Stokes wave grows from noise.
- Becomes efficient at relatively low pump powers.

Stimulated Brillouin Scattering

- Governed by two coupled equations:

$$\frac{dI_p}{dz} = -g_B I_p I_s - \alpha_p I_p, \quad -\frac{dI_s}{dz} = +g_B I_p I_s - \alpha_s I_s.$$

- Brillouin gain has a narrow Lorentzian spectrum:

$$g_B(\nu) = \frac{g_B(\nu_B)}{1 + 4(\nu - \nu_B)^2/(\Delta\nu_B)^2}.$$

- Phonon lifetime $T_B < 10$ ns results in gain bandwidth >30 MHz.
- Peak Brillouin gain $\approx 5 \times 10^{-11}$ m/W.
- Compared with Raman gain, peak gain larger by a factor of 1000 but its bandwidth is smaller by a factor of 100,000.
- SBS is the most dominant nonlinear process in silica fibers.

Growth of Stokes Wave

- Assume pump is so intense that its depletion can be ignored.
- Using solution for the pump beam, we obtain

$$dI_s/dz = -g_R(P_0/A_{\text{eff}})\exp(-\alpha_p z)I_s + \alpha_s I_s.$$

- Solution for a fiber of length L is given by

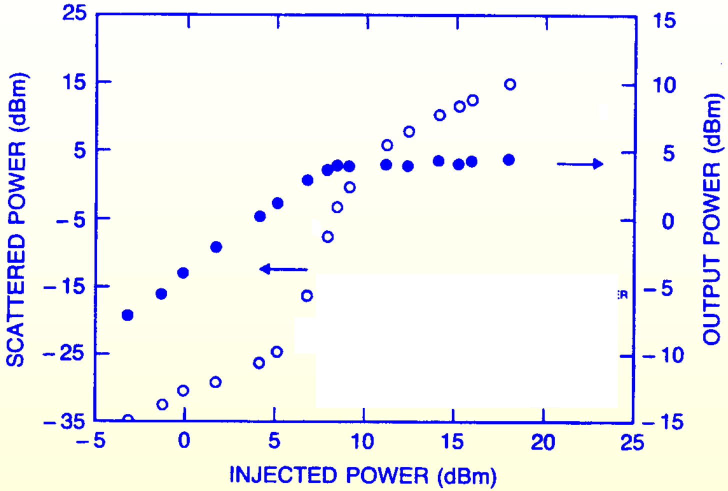
$$I_s(0) = I_s(L)\exp(g_B P_0 L_{\text{eff}}/A_{\text{eff}} - \alpha L),$$

- Stokes wave grows exponentially in the backward direction from an initial seed injected at the fiber output end at $z = L$.
- Threshold power P_{th} for SBS is found from

$$g_B(\nu_B)P_{\text{th}}L_{\text{eff}}/A_{\text{eff}} \approx 21.$$

- For long fibers P_{th} for the SBS onset can be as low as 1 mW.

Growth of Stokes Wave



- Transmitted (solid circles) and reflected (empty circles) powers as a function of input power for a 13-km-long fiber.
- Brillouin threshold is reached at a power level of about 5 mW.
- Reflected power increases rapidly after threshold and consists of mostly SBS-generated Stokes radiation.

SBS Threshold

- Optical signal in lightwave systems is in the form of a bit stream consisting of pulses whose width depend on the bit rate.
- Brillouin threshold higher for an optical bit stream.
- Calculation of Brillouin threshold quite involved because 1 and 0 bits do not follow a fixed pattern.
- Simple approach: Situation equivalent to that of a CW pump with a wider spectrum and a reduced peak power.
- Brillouin threshold increases by about a factor of 2 irrespective of the actual bit rate of the system.
- Channel powers limited to below 5 mW under typical conditions.

Control of SBS

- Some applications require launch powers in excess of 10 mW.
- An example provided by shore-to-island fiber links designed to transmit information without in-line amplifiers or repeaters.
- Threshold can be controlled by increasing either $\Delta\nu_B$ (about 30 MHz) or line width of the optical carrier ($<$10 MHz).
- Bandwidth of optical carrier can be increased by modulating its phase at a frequency lower than the bit rate (typically, $\Delta\nu_m < 1$ GHz).
- Brillouin gain is reduced by a factor of $(1+\Delta\nu_m/\Delta\nu_B)$.
- SBS threshold increases by the same factor.

Control of SBS

- Brillouin-gain bandwidth $\Delta\nu_B$ can be increased to more than 400 MHz by designing special fibers.
- Sinusoidal strain along the fiber length can be used for this purpose.
- Strain changes Brillouin shift ν_B by a few percent in a periodic manner.
- Strain can be applied during cabling of the fiber.
- Brillouin shift ν_B can also be changed by making core radius nonuniform along fiber length.
- Same effect can be realized by changing dopant concentration along fiber length.

Nonlinear Pulse Propagation

- Two analytic techniques can be used for solving the NLS equation approximately (the Moment method and Variational method).
- They can be used provided one can assume that the pulse maintains a specific shape inside the fiber link.
- Pulse parameters (amplitude, phase, width, and chirp) are allowed to change continuously with z.
- This assumption holds reasonably well in several cases of practical interest.
- A Gaussian pulse maintains its shape at low powers.
- Let us assume that the Gaussian shape remains approximately valid when the nonlinear effects are relatively weak.

Moment Method

- Treat the optical pulse like a particle whose energy E, RMS width σ, and chirp C are defined as $E = \int_{-\infty}^{\infty} |U|^2 dt$,

$$\sigma^2 = \frac{1}{E}\int_{-\infty}^{\infty} t^2 |U|^2 dt, \qquad C = \frac{i}{E}\int_{-\infty}^{\infty} t \left(U^* \frac{\partial U}{\partial t} - U \frac{\partial U^*}{\partial t} \right) dt.$$

- Differentiate them with respect to z and use the NLS Equation

$$\frac{\partial U}{\partial z} + \frac{i\beta_2}{2}\frac{\partial^2 U}{\partial t^2} = i\gamma P_0 p(z) |U|^2 U.$$

- We find that $dE/dz = 0$ but σ^2 and C satisfy

$$\frac{d\sigma^2}{dz} = \frac{\beta_2}{E}\int_{-\infty}^{\infty} t^2 \mathrm{Im}\left(U^* \frac{\partial^2 U}{\partial t^2} \right) dt,$$

$$\frac{dC}{dz} = \frac{2\beta_2}{E}\int_{-\infty}^{\infty} \left| \frac{\partial U}{\partial t} \right|^2 dt + \frac{\gamma P_0}{E} p(z) \int_{-\infty}^{\infty} |U|^4 dt.$$

Equations for Pulse Parameters

- For a chirped Gaussian pulse: $U(z,t) = a\,\exp[-\frac{1}{2}(1+iC)(t/T)^2]$.
- All four pulse parameters (a, C, T, and ϕ) are functions of z.
- Peak amplitude a is related to energy as $E = \sqrt{\pi}a^2T$.
- Width parameter T is related to the RMS width σ as $T = \sqrt{2}\sigma$.
- Width T and chirp C are found to change with z as

$$\frac{dT}{dz} = \frac{\beta_2 C}{T},$$

$$\frac{dC}{dz} = (1+C^2)\frac{\beta_2}{T^2} + \gamma P_0 \frac{p(z)}{\sqrt{2}}\frac{T_0}{T}.$$

- These two equations govern how the nonlinear effects modify the width and chirp of a Gaussian pulse.

Physical Interpretation

- Considerable physical insight gained from moment equations.
- SPM does not affect the pulse width directly as γ appears only in the chirp equation.
- Two terms on the right side of chirp equation originate from the dispersive and nonlinear effects, respectively.
- They have the same sign for normal GVD ($\beta_2 > 0$).
- Since SPM-induced chirp then adds to GVD-induced chirp, we expect SPM to increase the rate of pulse broadening.
- When GVD is anomalous ($\beta_2 < 0$), two terms have opposite signs, and pulse broadening should be reduced.
- Width equation leads to $T^2(z) = T_0^2 + 2\int_0^z \beta_2(z)C(z)\,dz$.

Variational Method

- Variational method uses the Lagrangian $\mathscr{L} = \int_{-\infty}^{\infty} \mathscr{L}_d(q, q^*)\, dt$.
- Lagrangian density $\mathscr{L}_d$ satisfies the Euler–Lagrange equation

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}_d}{\partial q_t}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \mathscr{L}_d}{\partial q_z}\right) - \frac{\partial \mathscr{L}_d}{\partial q} = 0,$$

- q_t and q_z are derivatives of q with respect to t and z, respectively.
- NLS equation can be derived from the Euler–Lagrange equation with $q = U^*$ when

$$\mathscr{L}_d = \frac{i}{2}\left(U^* \frac{\partial U}{\partial z} - U \frac{\partial U^*}{\partial z}\right) + \frac{\beta_2}{2}\left|\frac{\partial U}{\partial t}\right|^2 + \tfrac{1}{2}\gamma P_0 p(z)|U|^4.$$

- If pulse shape is known in advance, integration can be performed analytically to obtain $\mathscr{L}$ in terms of pulse parameters.

Lagrangian for a Gaussian Pulse

- In the case of a chirped Gaussian pulse

$$U(z,t) = a\exp[-\tfrac{1}{2}(1+iC)(t/T)^2 + i\phi].$$

- Lagrangian $\mathscr{L}$ is found to be

$$\mathscr{L} = \frac{\beta_2 E}{4T^2}(1+C^2) + \frac{\gamma p(z) E^2}{\sqrt{8\pi}T} + \frac{E}{4}\left(\frac{dC}{dz} - \frac{2C}{T}\frac{dT}{dz}\right) - E\frac{d\phi}{dz}.$$

- $E = \sqrt{\pi}a^2 T$ is the pulse energy.
- Final step: Minimize $\int \mathscr{L}(z)\,dz$ with respect to four pulse parameters using the Euler–Lagrange equation

$$\frac{d}{dz}\left(\frac{\partial \mathscr{L}}{\partial q_z}\right) - \frac{\partial \mathscr{L}}{\partial q} = 0.$$

- q represents one of the pulse parameters ($q_z = dq/dz$).

Variational Equations

- If we use $q = \phi$, we obtain $dE/dz = 0$.
- Using $q = E$, we obtain the phase equation:

$$\frac{d\phi}{dz} = \frac{\beta_2}{2T^2} + \frac{5\gamma p(z)E}{4\sqrt{2\pi}T}.$$

- Using $q = C$ and $q = T$, we obtain the width and chirp equations:

$$\frac{dT}{dz} = \frac{\beta_2 C}{T},$$

$$\frac{dC}{dz} = (1 + C^2)\frac{\beta_2}{T^2} + \gamma P_0 \frac{p(z)}{\sqrt{2}} \frac{T_0}{T}.$$

- These equations are identical to those obtained earlier with the moment method.

Specific Analytic Solutions

- Apply the moment equations to the linear case ($\gamma = 0$):

$$\frac{dT}{dz} = \frac{\beta_2 C}{T}, \qquad \frac{dC}{dz} = (1 + C^2)\frac{\beta_2}{T^2}.$$

- From dT/dC it follows that $(1+C^2)/T^2 = (1+C_0^2)/T_0^2$.
- General solution is found to be ($\xi = z/L_D$)

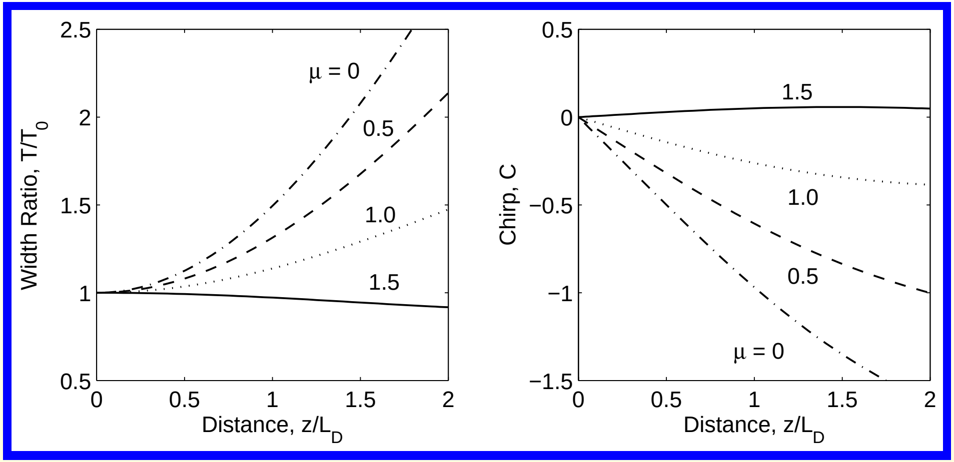
$$T^2(\xi) = T_0^2[1 + 2sC_0\xi + (1+C_0^2)\xi^2], \qquad C(z) = C_0 + s(1+C_0^2)\xi.$$

- These results agree with those obtained in Section 3.3.
- Assume nonlinear effects are weak and write as $C = C_L + C'$:

$$\frac{dC'}{dz} = \frac{\gamma P_0}{\sqrt{2}}\frac{T_0}{T}, \qquad C'(z) \approx \frac{\gamma P_0 T_0}{\sqrt{2}\beta_2 C_L}(T - T_0).$$

- Pulse width is found from $T^2(z) = T_0^2 + 2\beta_2 \int_0^z C(z)\,dz$.

Numerical Solution



- Numerical Solution for several values of $\mu = \gamma P_0 L_D$.
- The linear case corresponds to $\mu = 0$.
- Gaussian input pulses are assumed to be unchirped.
- As nonlinear effects increase, pulse broadens less and less and may even compress.

SPM-Induced Pulse Compression

- Pulse compression can be understood from the chirp equation:

$$\frac{dC}{dz} = (1+C^2)\frac{\beta_2}{T^2} + \gamma P_0 \frac{p(z)}{\sqrt{2}} \frac{T_0}{T}.$$

- In the case of normal GVD, two terms have the same sign.
- Pulse broadens even faster than that expected without SPM.
- Two terms have opposite signs when $\beta_2 < 0$.
- SPM cancels dispersion-induced chirp and reduces pulse broadening.
- For a certain value of $\mu = \gamma P_0 L_D$, two terms nearly cancel, and pulse width does not change (soliton formation).
- For larger values of μ, pulse would compress, at least initially.

Soliton Formation

- Use the moment method with $U(z,t) = a\,\text{sech}(t/T)\exp[-iC(t/T)^2]$.
- Width equation does not change: $\frac{dT}{dz} = \frac{\beta_2 C}{T}$.
- Chirp equation is modified slightly:

$$\frac{dC}{dz} = \left(C^2 + \frac{4}{\pi^2}\right)\frac{\beta_2}{T^2} + \gamma P_0 p(z)\frac{4}{\pi^2}\frac{T_0}{T}.$$

- Introducing $\tau = T/T_0$, $p(z) = 1$, and $L_D = T_0^2/|\beta_2|$:

$$L_D\frac{dC}{dz} = \left(C^2 + \frac{4}{\pi^2}\right)\frac{s}{\tau^2} + \gamma P_0 L_D \frac{4}{\pi^2\tau}.$$

- If initially $C = 0$ and $\tau = 1$, dC/dz remains 0 when $s = -1$ and peak power of the pulse satisfies $\gamma P_0 L_D = L_D/L_{\text{NL}} = 1$. Pulse maintains its width in spite of SPM and GVD (a soliton).