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Fraunhofer diffraction and the state of polarization of partially coherent electromagnetic beams

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We generalize the concept of Fraunhofer diffraction to partially coherent electromagnetic beams and show how the state of polarization is affected by a circular aperture. It is illustrated that the far-zone properties of a random beam can be tuned by varying the aperture radius. We find that even an incident beam that is completely unpolarized can sometimes produce a field that is highly polarized. ©2019 Optical Society of America

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Fraunhofer diffraction describes the far-zone properties of a field that has passed through an aperture in an opaque screen. Among its many applications are optical computing, image processing, and holography. In the classical description [1], the incident field is assumed to be both scalar and spatially fully coherent. The extension of Fraunhofer diffraction to partially coherent scalar fields has been considered in, for example, Refs. [2–5]. Those studies were all concerned with the effect of coherence on the intensity distribution. The influence of an aperture on the spectrum of the far-zone field was investigated in Refs. [6,7]. More recently, an analysis of the role of temporal coherence was presented in Ref. [8]. In many practical circumstances the state of polarization cannot be neglected, and a scalar description of the field does not suffice. Therefore, it is useful to extend the notion of Fraunhofer diffraction to stochastic electromagnetic beams. Here we present a theory that allows us to investigate how the far-zone state of polarization of such beams is affected by the diffraction process.

It is worth noting that when a random electromagnetic beam propagates, both its spectrum [9] and state of polarization may change [10], even if this propagation is through free space. These effects are referred to as “coherence-induced changes.”

As we will demonstrate, placing an aperture in the path of the beam will also change its far-zone properties.

Let us begin by considering a monochromatic, scalar wave with wavelength λ . If this wave is diffracted by an aperture in a plane, opaque screen, the resulting field in the far zone is proportional to the truncated Fourier transform of the incident field $U^{(in)}$ ([1], Eqs. (4)–(25)):

$$U^{(\infty)}(x, y, z) = \frac{e^{ikz}}{i\lambda z} e^{ik(x^2 + y^2)/2z} \int_{\mathcal{A}} U^{(in)}(\xi, \eta) e^{-i2\pi(x\xi + y\eta)/\lambda z} d\xi d\eta. \quad (1)$$

Here the screen is assumed to be in the xy plane, $k = 2\pi/\lambda$ is the wavenumber, and \mathcal{A} is the area of the aperture. The superscript ∞ denotes the far zone. We now let $U^{(in)}$ represent a Cartesian component of the incident electric field vector, i.e.,

$$U^{(in)}(\xi, \eta) = E_j^{(in)}(\xi, \eta), \quad (j = x, y). \quad (2)$$

The second-order statistical properties, at frequency ω , of a stochastic, electromagnetic beam at two positions \mathbf{r}_1 and \mathbf{r}_2 are characterized by the cross-spectral density matrix [11]

$$\mathbf{W}(\mathbf{r}_1, \mathbf{r}_2) = \begin{pmatrix} W_{xx} & W_{xy} \\ W_{yx} & W_{yy} \end{pmatrix}. \quad (3)$$

The four matrix elements are given by the expression

$$W_{ij}(\mathbf{r}_1, \mathbf{r}_2) = \langle E_i^*(\mathbf{r}_1) E_j(\mathbf{r}_2) \rangle \quad (i, j = x, y), \quad (4)$$

where the angular brackets indicate the average taken over an ensemble of beam realizations. On substituting from Eq. (1) into Eq. (4), and interchanging the order of integration and ensemble averaging, we find for the cross-spectral density matrix in a far-zone plane z the formula

$$W_{ij}^{(\infty)}(x_1, y_1, x_2, y_2; z) = \frac{1}{\lambda^2 z^2} \exp \left[\frac{ik}{2z} (x_2^2 + y_2^2 - x_1^2 - y_1^2) \right] \times \tilde{W}_{ij}^{(\text{in})} \left(-\frac{x_1}{\lambda z}, -\frac{y_1}{\lambda z}, \frac{x_2}{\lambda z}, \frac{y_2}{\lambda z} \right), \quad (5)$$

with the four-dimensional spatial Fourier transform defined as

$$\tilde{W}_{ij}^{(\text{in})}(p_1, q_1, p_2, q_2) = \int_{\mathcal{A}} \int_{\mathcal{A}} W_{ij}^{(\text{in})}(x_1, y_1, x_2, y_2) \times e^{-i2\pi(x_1 p_1 + y_1 q_1 + x_2 p_2 + y_2 q_2)} dx_1 dy_1 dx_2 dy_2. \quad (6)$$

Equation (5) states that each element of the cross-spectral density matrix in the far zone is, up to a position-dependent phase factor, related to its counterpart in the aperture by a Fourier transform. This formula generalizes the notion of Fraunhofer diffraction from monochromatic scalar fields to random electromagnetic beams.

If we restrict our attention to a far-zone observation point $Z = (0, 0, z)$ on the z axis, then Eq. (5) for two coincident points simplifies to

$$W_{ij}^{(\infty)}(Z, Z) = \frac{1}{\lambda^2 z^2} \int_{\mathcal{A}} \int_{\mathcal{A}} W_{ij}^{(\text{in})}(x_1, y_1, x_2, y_2) dx_1 dy_1 dx_2 dy_2. \quad (7)$$

It is seen that $W_{ij}^{(\infty)}(Z, Z)$ is proportional to the double integral of the corresponding element in the aperture. Because, as we will shortly discuss, the state of polarization depends only on \mathbf{W} at equal points, Eq. (7) has an important consequence. For a very small aperture, over which the spatial variation of the matrix elements may be neglected, we get that

$$W_{ij}^{(\infty)}(Z, Z) \approx \frac{d\mathcal{A}^2}{\lambda^2 z^2} W_{ij}^{(\text{in})}(0, 0, 0, 0), \quad \text{as } d\mathcal{A} \rightarrow 0, \quad (8)$$

where $d\mathcal{A}$ denotes the aperture area. This means that the normalized matrix elements at the far-zone point Z are identical to their counterparts at the center of the aperture. Therefore, *for a small enough aperture, the state of polarization at a far-zone point on the central axis is identical to the state of polarization of the field at the center of the aperture.*

We illustrate the usefulness of the above formalism by considering the example of an aperture field that is produced by a Gaussian Schell-model source [11]. If we assume the effective width of the two electric field components to be equal, we have that

$$W_{ij}^{(\text{in})}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = A_i A_j B_{ij} \exp \left[-\left(\frac{\rho_1^2 + \rho_2^2}{4\sigma^2} \right) \right] \exp \left[-\frac{(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1)^2}{2\delta_{ij}^2} \right], \quad (9)$$

where $\boldsymbol{\rho} = (x, y)$. The factors A_i denote the spectral amplitude of each Cartesian field component, B_{ij} is the complex-valued correlation coefficient between E_i and E_j , σ is the effective beam width, and δ_{ij} is a coherence radius. These parameters cannot be chosen arbitrarily, but have to satisfy several constraints. These follow from the definition of $\mathbf{W}^{(\text{in})}$ (Sec. 9.4.2, [11]), the realizability conditions [12], and the beam conditions [13].

We take the aperture to be a circle with radius a . On changing variables to

$$\boldsymbol{\rho}_1 = \rho_1 (\cos \phi_1, \sin \phi_1), \quad \boldsymbol{\rho}_2 = \rho_2 (\cos \phi_2, \sin \phi_2). \quad (10)$$

Equation (7) can be re-written as

$$W_{ij}^{(\infty)}(Z, Z) = \frac{A_i A_j B_{ij}}{\lambda^2 z^2} \int_0^a \int_0^{2\pi} \int_0^a \int_0^{2\pi} \rho_1 \rho_2 \exp \left(-\frac{\rho_1^2 + \rho_2^2}{4\sigma^2} \right) \times \exp \left(-\frac{\rho_1^2 + \rho_2^2}{2\delta_{ij}^2} \right) \times \exp \left[\frac{\rho_1 \rho_2 \cos(\phi_1 - \phi_2)}{\delta_{ij}^2} \right] d\phi_1 d\rho_1 d\phi_2 d\rho_2 \quad (11)$$

$$= \frac{4\pi^2 A_i A_j B_{ij}}{\lambda^2 z^2} \int_0^a \int_0^a \exp \left(-\frac{\rho_1^2 + \rho_2^2}{2\beta_{ij}^2} \right) \times \rho_1 \rho_2 I_0 \left(\frac{\rho_1 \rho_2}{\delta_{ij}^2} \right) d\rho_1 d\rho_2, \quad (12)$$

where I_0 denotes the modified Bessel function of order zero, and where we introduced the new parameter

$$\frac{1}{\beta_{ij}^2} = \frac{1}{\delta_{ij}^2} + \frac{1}{2\sigma^2}. \quad (13)$$

The limit of this expression, as the aperture radius $a \rightarrow \infty$, can be derived by considering the identity [14]

$$\int_0^\infty x e^{-x^2/(2b^2)} I_0(px) dx = b^2 e^{b^2 p^2/2}. \quad (14)$$

On making use of this in Eq. (12), we find that

$$W_{ij}^{(\infty)}(Z, Z) = \frac{4\pi^2 A_i A_j B_{ij}}{\lambda^2 z^2} \frac{\beta_{ij}^4 \delta_{ij}^4}{\delta_{ij}^4 - \beta_{ij}^4}, \quad \text{as } a \rightarrow \infty. \quad (15)$$

For a finite aperture radius, Eq. (12) is easily evaluated numerically. Having thus established the far-zone cross-spectral density matrix, the polarization properties of the field there can be expressed in terms of the four spectral Stokes parameters, denoted $S_n(Z)$, with $n = 0, 1, 2, 3$. Their expectation values are given by the formulas [11]

$$S_0(Z) = W_{xx}(Z, Z) + W_{yy}(Z, Z), \quad (16a)$$

$$S_1(Z) = W_{xx}(Z, Z) - W_{yy}(Z, Z), \quad (16b)$$

$$S_2(Z) = W_{xy}(Z, Z) + W_{yx}(Z, Z), \quad (16c)$$

$$S_3(Z) = i[W_{yx}(Z, Z) - W_{xy}(Z, Z)]. \quad (16d)$$

The normalized version of the Stokes parameters is defined as

$$s_m(Z) = S_m(Z)/S_0(Z), \quad (m = 1, 2, 3). \quad (17)$$

The degree of polarization (DOP), the ratio of the spectral density of the fully polarized part of the beam, and its total spectral density, follow from the relation (Sec. 6.3.3, [15])

$$\text{DOP}(Z) = \sqrt{s_1^2(Z) + s_2^2(Z) + s_3^2(Z)}. \quad (18)$$

The DOP is bounded by zero and one. The lower bound represents a completely unpolarized beam, whereas the upper bound represents a fully polarized beam. We next investigate how the state of polarization of the far-zone field, characterized

by the DOP and the Stokes parameters, can be tuned by varying the size of the aperture.

The dependence of the three normalized Stokes parameters, at the far-zone axial point Z , on the aperture radius a is shown in Fig. 1. The first parameter, $s_1(Z)$, remains constant at 0.6. Its value does not depend on a , because we have set $\delta_{xx} = \delta_{yy}$. The two other parameters, $s_2(Z)$ and $s_3(Z)$, increase significantly when the aperture radius a becomes larger. Notice that this tendency persists, even when a is larger than the effective beam width $\sigma = 7$ mm, idem dito for δ_{xx} , δ_{yy} , and δ_{xy} . Eventually, the parameters $s_2(Z)$ and $s_3(Z)$ tend to their asymptotic value which can be calculated from Eq. (15) and which is 0.30 and 0.52, respectively. Furthermore, as remarked below Eq. (8), in the limit $a \rightarrow 0$, the Stokes parameters are equal to their value at the center of the aperture. For example, it is easily verified from Eq. (9) and definition (18) that everywhere in the aperture $s_2 = 2A_x A_y |B_{xy}| \cos \phi / (A_x^2 + A_y^2) = 0.08$.

Similarly, the DOP is also strongly affected by the aperture size. This is illustrated in Fig. 2 in which $DOP(Z)$ is plotted for selected values of the amplitude A_x . Just as the Stokes parameters, the DOP on the far-zone axis is seen to be very sensitive to the aperture radius. As noted in connection with Fig. 1, this influence persists, even when $a > \sigma$. In addition, in agreement with the statement below Eq. (8), when the aperture radius is very small, i.e., $a < \sigma$, δ_{ij} , we find that $DOP(Z)$ is equal to its value at the center of the aperture.

Next, we consider an incident field that is completely unpolarized. One possible example (see also Ref. [16]) is a beam given by Eq. (9) while setting $A_x = A_y = A$, and $B_{xy} = B_{yx} = 0$. In that case,

$$W_{xx}^{(in)}(\rho, \rho) = W_{yy}^{(in)}(\rho, \rho) \neq 0, \quad (19)$$

$$W_{xy}^{(in)}(\rho, \rho) = W_{yx}^{(in)}(\rho, \rho) = 0. \quad (20)$$

From Eq. (12), it readily follows that the far-zone elements $W_{xy}^{(\infty)}(Z, Z) = W_{yx}^{(\infty)}(Z, Z) = 0$, and hence $s_2(Z) = s_3(Z) = 0$. However, if $\delta_{xx} \neq \delta_{yy}$, the two diagonal matrix elements, W_{xx} and W_{yy} , will evolve differently on propagation and, thus, give rise to a field that is partially polarized. An example is presented in Fig. 3. The larger the difference between

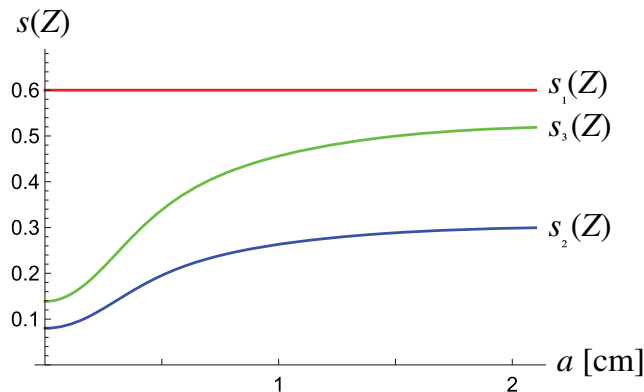


Fig. 1. Three normalized Stokes parameters on the far-zone axis as a function of the aperture radius a . The red curve corresponds to $s_1(Z)$, the blue curve = $s_2(Z)$, and the green curve = $s_3(Z)$. In this example $A_x = 2.0$, $A_y = 1.0$, $\sigma = 7$ mm, $\lambda = 632.8$ nm, $B_{xy} = 0.2e^{i\pi/3}$, $\delta_{xx} = \delta_{yy} = 2$ mm, and $\delta_{xy} = 4$ mm.

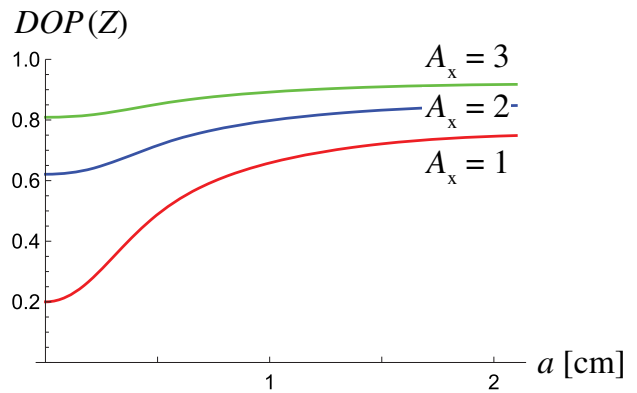


Fig. 2. DOP on the far-zone axis as a function of the aperture radius a , for selected values of A_x , the amplitude of E_x . The red curve is for $A_x = 1$, the blue curve is for $A_x = 2$, and the green curve is for $A_x = 3$. The amplitude A_y is kept fixed at 1, and all other parameters are as in Fig. 1.

δ_{xx} and δ_{yy} , the more $W_{xx}^{(\infty)}$ and $W_{yy}^{(\infty)}$ will diverge. This then causes the value of s_1 to increase, leading to a growing DOP. We emphasize that this effect does not occur for all unpolarized aperture fields, but hinges on the fact that the two Cartesian components of the electric field have different coherence radii, i.e., $\delta_{xx} \neq \delta_{yy}$.

We have restricted our analysis to the field on the far-zone axis. For non-axial points the more general expression Eq. (5) must be used. From the reciprocity between a function and its Fourier transform, we can get a qualitative idea of the off-axis field. As the width of the elements $W_{ij}^{(in)}$ gets less, their far-field counterparts will become broader, and the off-axis and the on-axis state of polarization will be more similar.

In conclusion, we have extended the concept of Fraunhofer diffraction from scalar fields to stochastic electromagnetic beams. This allowed us to study the polarization properties of the on-axis field in the far zone. The Stokes parameters and the DOP were both found to be quite sensitive to the

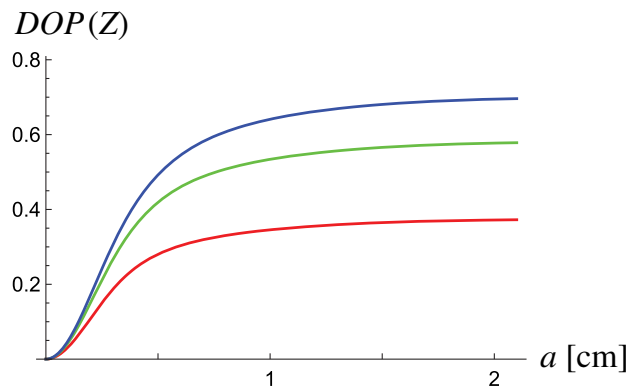


Fig. 3. Far-zone DOP produced by a completely unpolarized incident beam, as a function of the aperture radius a , for selected values of δ_{xx} , the coherence radius of E_x . The lower (red) curve is for $\delta_{xx} = 3$ mm, the middle (green) curve is for $\delta_{xx} = 4$ mm, and the top (blue) curve is for $\delta_{xx} = 5$ mm. Here $A_x = A_y$, $\sigma = 7$ mm, $\delta_{yy} = 2$ mm, and $\lambda = 632.8$ nm.

aperture size. An example was given in which an incident beam that is completely unpolarized becomes highly polarized in the far zone. The enhancement of the DOP is due to the fact that, in general, the four elements of the cross-spectral density matrix evolve differently on propagation. If, for example, W_{xx} becomes dominant, the beam will become essentially x polarized. This enhancement of the DOP will be useful for applications that use incoherent illumination, but require some coherence in the far field. An example would be where the x and y polarized components are forced to interfere after passing them through a polarizer oriented at 45 deg.

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