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# Invite paper: Self-imaging in multimode graded-index fibers and its impact on the nonlinear phenomena



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Keywords: Nonlinear optics Multimode fibers Graded index Self-imaging	The phenomenon of periodic self-imaging of optical beams, occurring inside any graded-index (GRIN) medium, was studied during the decade of the 1970s and was exploited to commercialize the GRIN lens. It has been found in recent years that the periodic self-imaging also affects the nonlinear propagation of optical pulses inside multimode GRIN fibers. In this paper, we first present the theory of self-imaging in linear GRIN fibers using a modal expansion approach. It is shown that the optical field at any point inside the fiber can be written without any reference to the fiber modes as a two-dimensional integration over the input field using a propagation kernel that is similar to that found in diffraction theory. However, this kernel has a specific property that reproduces the input field precisely in a periodic fashion along the length of a GRIN fiber (self-imaging). We apply this kernel to study the propagation of a Gaussian beam and discuss how self-imaging is modified by self-focusing produced by the Kerr nonlinearity. We then consider propagation of the continuous and pulsed Gaussian beams inside a GRIN fiber and discuss how self-imaging affects the modulation instability, leads to the formation of GRIN solitons, and produces novel temporal and spectral features when short optical pulses are launched that are intense enough to form high-order solitons.

# 1. Introduction

Light propagation in graded-index (GRIN) media was investigated during the 1970s, motivated mostly by their applications in optical communication systems [1–6]. By the year 1980, smaller differential modal delays of GRIN fibers compared to step-index fibers led to the use of GRIN fibers in the first-generation communication systems. The interest in GRIN fibers declined after 1985 as the use of single-mode fibers became dominant. It was only after 2010 that multimode fibers attracted renewed attention for enhancing the capacity of optical communication systems through the technique of space-division or modedivision multiplexing [7–9]. This revival has led to a resurgence of interest in GRIN fibers, especially in their nonlinear properties [10–15]. Among the nonlinear phenomena that have attracted attention are soliton formation inside GRIN fibers [11,16], geometric parametric instability [12], and spatial-beam cleanup [13].

The phenomenon of self-imaging, a well-known property of any GRIN medium [2,6], has been found to play a crucial role in these nonlinear studies [13]. In this paper we first review in Section 2 the optical modes supported by a GRIN fiber. We then develop in Section 3 theory behind self-imaging using a modal-expansion approach and show that the output field at any point inside the fiber can be obtained, without any reference to the fiber modes, using a propagation kernel

that is similar to one found in diffraction theory. However, this kernel has a specific property such that any input field is reproduced precisely in a periodic fashion along the length of a GRIN fiber (self-imaging). We apply this kernel in Section 4 to study the propagation of a Gaussian beam inside a GRIN fiber and recover a known result for its periodically varying beam width. We also discuss how self-imaging is modified by self-focusing produced by the Kerr nonlinearity. In Section 5 we consider propagation of a pulsed Gaussian beam inside a GRIN fiber and derive an effective nonlinear Schrödinger equation that includes the effects of periodic self-imaging. We also discuss the conditions under which such an equation can be used. In Section 6 we use this equation to study how self-imaging affects the phenomenon of modulation instability. The formation of GRIN solitons is covered in Section 7. We incorporate the high-order dispersive and nonlinear effects in Section 8 and discuss the novel temporal and spectral features occurring when femtosecond optical pulses that are intense enough to form high-order solitons are launched into a GRIN fiber.

# 2. Modes of GRIN fibers

In this section we consider the propagation of a continuous-wave (CW) beam of frequency  $\omega$  inside a GRIN fiber. The refractive index of most GRIN fibers decreases radially inside the core of radius *a* from its

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value  $n_1$  at the center to the cladding index  $n_c$  as [17]

$$n^{2}(x, y) = n_{1}^{2} [1 - 2\Delta(\rho/a)^{2}], \quad \rho = \sqrt{x^{2} + y^{2}}.$$
 (1)

The parameter  $\Delta = (n_1 - n_c)/n_1$  plays an important role and is defined in the same way as for step-index fibers [18]. The modes of GRIN fibers are obtained by solving the Helmholtz equation

$$\nabla^2 \mathbf{E} + n^2(x, y) k_0^2 \mathbf{E} = 0.$$
<sup>(2)</sup>

where  $k_0 = \omega/c$  at the optical frequency  $\omega$ . Although a numerical approach is necessary in general, this equation can be solved analytically if we assume that the index profile in Eq. (1) applies for all values of  $\rho$ .

In the so-called weakly guiding approximation ( $\Delta \ll 1$ ), both the electric and magnetic fields of all modes lie in a plane transverse to the fiber's axis ( $E_z = H_z = 0$ ), and the modes are denoted as  $LP_{mn}$ , where m and n are two integers used for labeling different modes. Their modal distribution  $F_{mn}(x, y)$  and propagation constants  $\beta_{mn}$  are known, but have different forms depending on whether Eq. (2) is solved using the Cartesian or cylindrical coordinates. We refer to Ref. [17] for their expressions in the two coordinate systems. As an example, the modal propagation constants in the Cartesian coordinates are given by

$$\beta_{mn} = n_1 k_0 \left[ 1 - \frac{2(m+n-1)}{n_1 k_0 a} \sqrt{2\Delta} \right]^{1/2}.$$
(3)

For most GRIN fibers,  $k_0 a \ge 1$  and  $\Delta \le 0.01$ . As a result, as long as m + n is not too large, we can expand  $\beta_{mn}$  in a binomial series and approximate it as

$$\beta_{mn} \approx n_1 k_0 - (m+n-1)\sqrt{2\Delta}/a. \tag{4}$$

This equation reveals the most important feature of the modes of a GRIN fiber. It shows that the propagation constants of all non-degenerate modes form a ladder-like structure with equal spacing between any two neighboring modes. This feature is similar to the quantized energy levels of a harmonic oscillator and is the physical mechanism behind the self-imaging phenomenon in GRIN fibers. In the following discussion, we assume that all modes have been normalized such that

$$\int \int_{-\infty}^{\infty} F_{mn}\left(x, y\right) F_{m'n'}^{*}\left(x, y\right) dx \, dy = \delta_{mm'}\delta_{nn'},\tag{5}$$

where the integration is over the entire transverse plane.

#### 3. Self-imaging theory

Any optical beam with the input field E(x, y, 0), in general, excites multiple fiber modes such that

$$E\left(x, y, 0\right) = \sum_{m} \sum_{n} c_{mn} F_{mn}\left(x, y\right),$$
(6)

where the sum extends over the whole range of the two integers  $(m, n = 0 \text{ to } \infty)$ . We can find the expansion coefficients  $c_{mn}$  by multiplying Eq. (6) with  $F_{m'n'}^*(x, y)$ , integrating over the whole transverse plane, and using the mode-orthogonality relation in Eq. (5). The result is given by

$$c_{mn} = \int \int_{-\infty}^{\infty} F_{mn}^* \left( x, y \right) E\left( x, y, 0 \right) dx \, dy.$$
<sup>(7)</sup>

The optical field E(x, y, z) at any point inside the GRIN fiber is obtained by multiplying each mode with a phase factor such that

$$E\left(x, y, z\right) = \sum_{m} \sum_{n} c_{mn} F_{mn}\left(x, y\right) \exp\left(i\beta_{mn}z\right).$$
(8)

Substituting  $c_{mn}$  from Eq. (7), we can write the result in the form

$$E\left(x, y, z\right) = \int \int_{-\infty}^{\infty} K\left(x, x'; y, y'\right) E\left(x', y', 0\right) dx' dy', \tag{9}$$

where the propagation kernel is given by

$$K\left(x, x'; y, y'\right) = \sum_{m} \sum_{n} F_{mn}\left(x, y\right) F_{mn}^{*}\left(x', y'\right) e^{i\beta_{mn}z}.$$
(10)

Eq. (9) provides the optical field at any point inside the GRIN medium in terms of the known input field at z = 0. Its form is similar to that used for diffraction of optical beams inside a homogeneous medium of constant refractive index. If the double sum in Eq. (10) can be evaluated in a closed form, the resulting kernel will include all of the excited modes of a GRIN fiber, without any explicit reference to them. It turns out that the double sum can be performed for GRIN fibers because of the ladder-like structure of the modal propagation constants. We refer to a 1974 paper for details [2] and write the result directly

$$K\left(x, x'; y, y'\right) = \frac{\beta}{2\pi i} \left(\frac{b}{\sin(bz)}\right) e^{i\psi}$$
$$\exp\left[\frac{i}{2}\beta b\cot(bz) \left(x'^{2} + {y'}^{2}\right) - \frac{i\beta b}{\sin(bz)} \left(xx' + yy'\right)\right],$$
(11)

where  $\beta = n_1 k_0$ ,  $b = \sqrt{2\Delta}/a$ , and the phase  $\psi$  depends only on the location of the point  $\mathbf{r} = (x, y, z)$  as

$$\psi(\mathbf{r}) = \beta z + \frac{1}{2}\beta b \cot(bz) \left(x^2 + y^2\right).$$
(12)

Noting that  $\sin(bz)/b = z$  in the limit  $b \to 0$ , it is easy to see that the kernel in Eq. (11) reduces to its form expected for a homogeneous medium of constant refractive index in that limit.

The kernel in Eq. (11) takes a somewhat simpler form in the cylindrical coordinates  $\rho$  and  $\phi$ :

$$K\left(\rho, \rho'; \phi, \phi'\right) = \frac{\beta}{2\pi i} \left(\frac{b}{\sin(bz)}\right) e^{i\beta z}$$
$$\exp\left[\frac{i}{2}\beta b \cot(bz) \left(\rho^{2} + {\rho'}^{2}\right) - \frac{i\beta b \rho \rho'}{\sin(bz)} \cos\left(\phi - \phi'\right)\right].$$
(13)

This form makes it easier to conclude that a radially symmetric input field (with no explicit dependence on  $\phi'$ ) will maintain this symmetry as it propagates down a GRIN fiber. To see this, we first write Eq. (9) in the cylindrical coordinates as

$$E\left(\rho,\,\phi,\,z\right) = \int_0^\infty \int_0^{2\pi} K\left(\rho,\,\rho';\phi,\,\phi'\right) E\left(\rho',\,\phi',\,0\right) \rho d\rho'\,d\phi'.$$
(14)

If  $E(\varphi', \phi', 0)$  does not depend on  $\phi'$ , the integration over  $\phi'$  can be carried out using the known result

$$\int_{0}^{2\pi} \exp\left[-ip\cos\left(\phi - \phi'\right)\right] d\phi' = 2\pi J_0(p).$$
(15)

As this integral does not depend on  $\phi$ , initial radial symmetry of any input beam is maintained during its propagation inside a GRIN fiber. In the modal picture, only the radially symmetric modes are excited by such an input beam.

The self-imaging property of a GRIN medium follows from the observation that the Kernel in Eq. (11) is reduced to the form

$$K(x, x'; y, y') = \delta(x - x')\delta(y - y')e^{i\beta z}$$
(16)

at distances that are an integer multiple of the period  $2\pi/b$ . This can be seen by noting that  $\cot(bz)$  can be replaced with  $1/\sin(bz)$  at such distances and *K* in Eq. (11) can be written as  $K = f(x - x')f(y - y')e^{ikz}$ , where the function f(x) is defined as

$$f(x) = \sqrt{\frac{q}{\pi}} \exp\left(-qx^2\right), \qquad q = \frac{\beta b}{2i\sin(bz)}.$$
(17)

It is easy to show that  $\int_{-\infty}^{\infty} f(x) dx = 1$ . At distances  $z = 2m\pi/b$  where *m* is an integer, *q* becomes infinitely large, and f(x) is reduced to the delta function  $\delta(x)$ . It follows from Eqs. (9) and (16) that the field  $E(\mathbf{r})$  becomes identical to the input field at all such distances, resulting in self-imaging.

Self-imaging also occurs at a shorter distance  $z_p = \pi/b$  with one major difference. In this case, one can show that the delta functions in Eq. (16) are replaced with  $\delta(x + x')$  and  $\delta(y + y')$ . As a result,  $E(x, y, z_p) = E(-x, -y, 0)$ , i.e., the image is flipped in both transverse directions. If the input field is radially symmetric, the sign changes have no impact, and the input field is reproduced (self-imaging) for the first time at the distance  $z_p$  and then periodically at distances that are multiples of  $z_p$ . It is important to stress that self-imaging at the distance  $2z_p$  occurs for an arbitrary input field, without any restriction on its functional form. This is the reason why GRIN rods can be used as a lens. We refer to a 1976 paper for further details on the imaging characteristics of a GRIN medium [4]. In particular, it can be shown that the ratio  $f = \cot(bz)/b$  plays the role of the focal length of such GRIN lenses. Selfimaging can occur even when the input beam is only partially coherent [19].

#### 4. Self-imaging of a CW Gaussian beam

A Gaussian-shape input beam is often launched into a GRIN fiber. It is thus useful to apply Eq. (9) to an input beam for which

$$E\left(x, y, 0\right) = A_0 \exp\left(-\frac{x^2 + y^2}{2w_0^2}\right),$$
(18)

where  $A_0$  is the peak amplitude and  $w_0$  is the spot size (1/e width) of the Gaussian beam. The full width at half-maximum (FWHM) of the beam is related to  $w_0$  as  $w_f = 1.665w_0$ . We can find the electric field at any point  $\mathbf{r} = (x, y, z)$  inside the fiber by using Eq. (18) in Eq. (9) together with the kernel in Eq. (11). The two integrations can be performed by using the known integral

$$\int_{-\infty}^{\infty} \exp\left(-px^2 + qx\right) dx = \sqrt{\pi/p} \exp\left(q^2/4p\right).$$
(19)

The final result can be written as

$$E(\mathbf{r}) = A_0 F(\mathbf{r}) \exp[i\phi(\mathbf{r})], \qquad (20)$$

where the beam shape is governed by

$$F(\mathbf{r}) = \frac{w_0}{w(z)} \exp\left[-\frac{(x^2 + y^2)}{2w^2(z)}\right]$$
(21)

and the spatial width w(z) evolves with z in a periodic fashion as

$$w(z) = w_0 \sqrt{\cos^2(\pi z/z_p) + C^2 \sin^2(\pi z/z_p)}.$$
(22)

The spatial period  $z_p$  and parameter *C* are defined as

$$z_p = \frac{\pi a}{\sqrt{2\Delta}}, \qquad C = \frac{z_p/\beta}{\pi w_0^2},$$
(23)

The phase  $\phi(\mathbf{r})$  in Eq. (20) is also spatially varying and has the form

$$\phi(\mathbf{r}) = \frac{\beta}{2w} \frac{dw}{dz} \left( x^2 + y^2 \right) + \beta z + \tan^{-1} \left( C \tan b z \right).$$
(24)

Eq. (21) shows that a Gaussian beam maintains its Gaussian shape inside a GRIN fiber, but its amplitude and width (also phase) change in a periodic fashion such that the beam recovers all of its input features periodically at distances  $z = mz_p$  (*m* is a positive integer) because of the self-imaging phenomenon. At distances  $z = (m + \frac{1}{2})z_p$ , the beam's width takes its minimum value  $w_0C$ , i.e., *C* governs the extent of beam



**Fig. 1.** Evolution of a Gaussian beam inside a GRIN fiber over two self-imaging periods for C = 0.5.

compression during each cycle. As an example, Fig. 1 shows the evolution of a Gaussian beam over two periods along the fiber's length using C = 0.5. For this value of C, the beam width is reduced by a factor of two at the point of maximum compression and its peak intensity is enhanced by a factor of 4. Much more intensity enhancement can occur for shorter values of C. Note also that the phase front becomes curved as the beam propagates down the fiber (spatial chirping), and its curvature also displays a periodic behavior. In particular, the phase front becomes planar at distance that are multiples of  $z_p/2$ . This is evident from Eq. (24) if we note that the slope dw/dz = 0 at distances where pulse width attains its minimum or maximum value.

It is important to estimate the values of two parameters defined in Eq. (23). Using  $\Delta = 0.01$  with  $a = 25 \,\mu\text{m}$  (typical values for commercial GRIN fibers), we find  $z_p = 0.55 \,\text{mm}$ , a remarkably short distance at which self-imaging first occurs inside such a GRIN fiber. Assuming  $w_0 = 8 \,\mu\text{m}$  and using  $\beta = 2\pi n_1/\lambda$  with  $n_1 = 1.45$  and  $\lambda = 1.06 \,\mu\text{m}$ , we find  $C \approx 0.3$ , indicating that the beam width is reduced to 30% of its initial value at  $z_p/2$ , before it recovers its input value at a distance of  $z_p$ . Even smaller values of *C* can be realized in practice by increasing the initial spot size  $w_0$  of the Gaussian beam. Fig. 2 shows how the ratio  $w/w_0$  varies over one self-imaging period for several values of the parameter *C*.

So far, we have neglected the nonlinear effects by assuming that the input power of the CW beam was low enough that all such effects were negligible. One may ask what happens to the self-imaging property of GRIN fibers when input power becomes large enough that the Kerr nonlinearity of the silica material must be considered. It is known that the Kerr contribution  $n_2I$  to the refractive index can lead to self-focusing of an optical beam with intensity I even inside a homogeneous medium of constant refractive index. At a critical power level,  $P_{cr}$ , self-focusing becomes catastrophic in the sense that the beam width shrinks to zero at a finite distance given by [20]



**Fig. 2.** Beam-width ratio  $w/w_0$  plotted over one self-imaging period for several values of the *C* parameter.

As a GRIN medium also reduces the beam size, the two effects may act together in such a way that catastrophic self-focusing occurs even at a shorter distance. Clearly, self-focusing can destroy the self-imaging property when input power is close to  $P_{\rm cr}$ .

The real question is whether self-imaging occurs when input power is well below  $P_{cr}$  but the nonlinear effects cannot be ignored. This question was answered in 1992 by solving the Gaussian-beam propagation problem with the variational technique [21], after adding the nonlinear contribution  $n_2I$  to the refractive index n(x, y) in Eq. (1). It was found that the beam width oscillates as indicated in Eq. (22) with same period  $z_p$  but the parameter *C* in Eq. (23) is modified as

$$C = \sqrt{1 - (P/P_{\rm cr})} \left( \frac{z_p / \beta}{\pi w_0^2} \right).$$
 (26)

The Kerr nonlinearity reduces the C parameter, but it does not affect the period of self-imaging. In physical terms, the Kerr nonlinearity only enhances the extent of beam compression during each self-imaging cycle. As long as the input power of a CW beam remains below the critical level of self-focusing, periodic self-imaging occurs just as it would in the absence of the nonlinear effects.

### 5. Nonlinear pulse propagation

In this section we consider propagation of a pulsed Gaussian beam inside a nonlinear GRIN fiber. The full problem is quite complicated because it requires numerical solutions of a nonlinear wave equation involving four variables (x, y, z, and t). A modal approach is often used in practice [22]. Its use requires solving many coupled equations with a large number of nonlinear terms and is limited in practice to fibers supporting a relatively small number of modes. It was found in 2017 that a simpler approach is possible for multimode GRIN fibers [14]. Its main assumption is that the bandwidth of the pulse is narrow enough that the spatial profile  $F(\mathbf{r})$  of the beam does not vary much over this bandwidth. By exploiting the self-imaging property of such fibers, we can write the electric field in the frequency domain as

$$\widetilde{E}(\mathbf{r},\,\omega) = \widetilde{A}(z,\,\omega)F(\mathbf{r})e^{i\beta(\omega)z},\tag{27}$$

where  $F(\mathbf{r})$  is at the center frequency  $\omega_0$  of the pulse and  $\beta(\omega)$  is the frequency-dependent propagation constant. It is important to keep in mind that  $F(\mathbf{r})$  is not the spatial profile of a specific mode but results from a superposition of all the modes excited by the input beam. This is reflected through the periodic evolution of  $F(\mathbf{r})$  along the fiber's length as indicated in Eq. (21).

As is often done in practice [18], we expand  $\beta(\omega)$  in a Taylor series around a center frequency  $\omega_0$  as

$$\beta(\omega) = \beta_0 + \beta_1(\omega - \omega_0) + \beta_2(\omega - \omega_0)^2 / 2 + \cdots,$$
(28)

where the dispersion parameters  $\beta_m = (d^m \beta/d\omega^m)$  are evaluated at the center frequency  $\omega = \omega_0$ . After eliminating the transverse coordinates through a spatial integration and going back to the time domain, the amplitude A(z, t) is found to satisfy [14]

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + \frac{i\beta_2}{2} \frac{\partial^2 A}{\partial t^2} = i\overline{\gamma}(z) \left| A \right|^2 A, \tag{29}$$

where the nonlinear parameter is defined as

$$\overline{\gamma}(z) = \frac{\omega_0 n_2}{cA_{\text{eff}}(z)}, \quad A_{\text{eff}}(z) = \left( \iint \left| F(\mathbf{r}) \right|^4 dx \, dy \right)^{-1}.$$
(30)

Recall that  $F(\mathbf{r})$  is normalized such that  $\iint |F(\mathbf{r})|^2 dx dy = 1$ . We call  $A_{\text{eff}}(z)$  the *effective beam area* to distinguish it from the *effective mode area*, whose value remains constant with z in single-mode fibers [18]. Eq. (29) with a constant value of  $\gamma$  is known as the nonlinear Schrödinger (NLS) equation. The group-velocity dispersion (GVD) of the fiber is included through the parameter  $\beta_2$ , which can be positive or negative at the center frequency  $\omega_0$  of the pulse.

Eq. (29) shows that the pulse evolution inside a GRIN fiber can be studied by solving a single NLS equation, even though multiple spatial modes may be propagating simultaneously inside the fiber. The oscillating spatial width of a Gaussian beam, resulting from the GRIN nature of the fiber and its self-imaging property, gives rise to an effective nonlinear parameter  $\overline{\gamma}(z)$  that is periodic in *z*. This is not surprising since the intensity at a given distance *z* depends on the beam width, becoming larger when the beam compresses and smaller as it spreads (see Fig. 1). One can also interpret the same effect as a periodically varying effective beam area. The spatial integrals appearing in Eq. (30) can be performed analytically using the functional form of  $F(\mathbf{r})$  in Eq. (21). The result can be written in the form  $\overline{\gamma}(z) = \gamma/f(z)$ , where  $\gamma$  is defined using the initial value of  $A_{\text{eff}}$  at z = 0. As a final step, if we use the reduced time  $T = t - \beta_1 z$ , Eq. (29) takes the form

$$\frac{\partial A}{\partial z} + \frac{i\beta_2}{2} \frac{\partial^2 A}{\partial T^2} = i\gamma f^{-1}(z) \left| A \right|^2 A, \tag{31}$$

where the function f(z) is found from Eq. (22) and has the form

$$f(z) = \cos^2(\pi z/z_p) + C^2 \sin^2(\pi z/z_p).$$
(32)

The modified NLS Eq. (31) involves only two variable (z and T) and can be solved numerically much faster than the full four-dimension problem for GRIN fibers. It includes the effects of a spatially evolving beam through the function f(z) and the two parameters C and  $z_p$  appearing in it. However, it neglects the impact of temporal dynamics on the spatial features of the beam and cannot be applied under all experimental conditions. It is thus important to summarize the conditions under which it can be used in practice.

- Weakly guiding approximation requiring  $\Delta \ll 1$  must hold. This approximation is often valid in practice as  $\Delta < 0.01$  for most GRIN fibers.
- The core size of the GRIN fiber must be large enough for it to support a large number of modes. As the core radius *a* > 20μm for most GRIN fibers, this is often the case in practice.
- Spatial width of the input beam must be smaller than 2*a* but still large enough that many low-order modes are excited simultaneously.
- At the same time, the total number of excited modes should be considerably less than 1000 to ensure that the approximation in Eq. (4) holds.
- The spectral bandwidth of pulses must be such that the beam shape does not change much over its entire range.

#### 6. Modulation instability

Modulation instability is a well-known nonlinear phenomenon in the context of single-mode optical fibers [18]. Occurring in the anomalous-GVD region of such fibers, it produces temporal oscillations on a CW beam that are reshaped by the Kerr nonlinearity into a train of optical solitons. It is thus natural to wonder how this instability is affected by the multimode nature of optical fibers and whether it can occur in the region of normal GVD of a GRIN fiber. To answer such questions, a linear stability analysis of the CW solution must be carried out to find the frequencies of the sidebands that appear on both sides of the narrow-width spectrum of the CW beam and grow with the gain provided by this instability.

The stability of a CW Gaussian beam inside a GRIN fiber was studied [23] in 2003 using the analytic solution given in Eq. (20). A more general analysis was carried out in 2019 in terms of a Hill's equation [24]. In both cases, the analysis is quite complicated. The modified NLS Eq. (31) provides a simpler approach for studying modulation instability in GRIN fibers [14]. In this case, the stability of the CW solution of Eq. (31) is analyzed by using

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$$A = \left(\sqrt{P_0} + a\right) \exp\left(i\phi_{\rm NL}\right), \quad \phi_{\rm NL} = \gamma P_0 \int_0^z f^{-1}(z) dz, \tag{33}$$

where  $P_0 = A_0^2$  is the input peak power and a(z, t) is a small perturbation. Linearizing Eq. (31) in *a*, we obtain the following equation:

$$\frac{\partial a}{\partial z} = -\frac{i\beta_2}{2}\frac{\partial^2 a}{\partial T^2} + \frac{i\gamma P_0}{f(z)}\left(a + a^*\right).$$
(34)

We seek its solution in the form  $a = a_1 e^{-i\Omega T} + a_2 e^{i\Omega T}$ , where  $\Omega$  is the modulation frequency and the amplitudes  $a_1$  and  $a_2$  satisfy:

$$\frac{da_k}{dz} = \frac{i}{2}\beta_2 \Omega^2 a_k + \frac{i\gamma P_0}{f(z)} \left( a_k + a_{3-k}^* \right).$$
(35)

where k = 1 or 2. These two coupled equations must be solved to find the frequencies for which the CW solution in Eq. (20) becomes unstable.

Because of the periodic nature of f(z), the preceding coupled equations can be solved approximately by expanding  $f^{-1}(z)$  in a Fourier series as  $f^{-1}(z) = \sum c_m e^{imKz}$ , where  $K = 2\pi/z_p$  and  $z_p$  is the self-imaging period given in Eq. (23). The coefficients  $c_m$  in this series are calculated using

$$c_m = \frac{1}{z_p} \int_0^{z_p} f^{-1}(z) e^{-imKz} dz,$$
(36)

Each term in the Fourier series helps in satisfying a phase-matching condition and results in one pair of sidebands located on opposite sides of the pump's frequency. The frequencies at which the gain of each sideband pair becomes maximum are found to satisfy [25]

$$\Omega_m^2 = \frac{2\pi m}{\beta_2 z_p} - \frac{2c_0}{\beta_2 L_{\rm NL}},\tag{37}$$

where *m* is an integer and the nonlinear length  $L_{\rm NL} = (\gamma P_0)^{-1}$ . The peak gain of the *m*th sideband depends on the Fourier coefficient  $c_m$  and is given by  $g_m = 2\gamma P_0 |c_m|$ . It is easy to show that  $c_0 = 1/C$ . The other Fourier coefficients can also be computed in terms of *C* [24]. The m = 0 sideband corresponds to the single-mode case and exists only if  $\beta_2 < 0$ , i.e., when GVD of the fiber is anomalous at the pump wavelength.

Eq. (37) shows that a CW Gaussian beams whose width oscillates along a GRIN fiber can become unstable to small perturbations, even when it propagates in the normal-GVD region of the fiber. The gain spectrum of modulation instability exhibits a rich structure with an infinite number of sideband pairs at frequencies that are not equally spaced. The peak gain of each sideband depends on the spatial pattern of the oscillating Gaussian beam through f(z). Since spatial variations play a crucial role, this instability is also known as a geometric parametric instability [12].

The numerical values of the sideband frequencies can be estimated using typical values for commercial GRIN fibers. We saw earlier that the self-imaging period  $z_p$  is < 1 mm in such fibers. As  $\gamma$  is relatively small for silica fibers (< 0.1 W<sup>-1</sup>/km), the nonlinear length exceeds 1 meter even at input power levels as high as 10 kW. The estimated value of  $c_0$  is < 5 for  $w_0 \approx 10 \,\mu$ m. As a result, the first term in Eq. (37) dominates in most cases of practical interest, and we can approximate the sideband frequencies in the case of normal GVD as

$$f_m = \Omega_m / 2\pi \approx \pm \sqrt{m/(2\pi\beta_2 z_p)}, \quad m = 1, 2, ....$$
 (38)

As a specific example, using  $\beta_2 = 20 \text{ ps}^2/\text{km}$  and  $z_p = 0.5 \text{ mm}$ , the *m*th sideband is found to be shifted by  $125\sqrt{m}$  THz from the pump frequency. For a pump laser with its center frequency close to 300 THz, the m = 1 sidebands will be located at frequencies near 175 THz and 425 THz. The former is in the infrared region (1700 nm), while the latter lies in the visible region near 700 nm. Moreover, the higher-order sidebands would fall in the ultraviolet and mid-infrared regions. Note that all sidebands exist even when  $\beta_2 < 0$  at the pump wavelength because the integer *m* can take negative values in Eq. (37). In this case frequency shift for the m = 0 sideband pair is typically below 1 THz.



Fig. 3. (a) Experimental (bottom) and numerically simulated (top) spectra when a 6-m-long GRIN fiber is pumped with 900-ps pulses with 50 kW peak power. Dashed vertical lines and circled numbers show the peak locations predicted by Eq. (38). (After Ref. [12]; ©2016 APS.)

The sideband frequencies predicted by Eq. (38) were seen in a 2016 experiment in which relatively long (900 ps) pulses at 1064 nm were launched into a 6-m-long GRIN fiber to mimic a quasi-CW situation [12]. The bottom trace in Fig. 3 shows the experimentally observed spectrum when the peak power of input pulses was 50 kW. The top trace shows the numerically simulated spectrum. Dashed vertical lines mark the locations of the peaks predicted by Eq. (38). The agreement seen in this figure indicates the usefulness of the modified NLS Eq. (31). Its only drawback is that it cannot capture spatial changes that may occur in response to temporal changes.

# 7. GRIN solitons

The question whether solitons can form inside multimode fibers attracted attention during the 1980s [26,27]. It was realized that different group delays (or speeds) associated with different modes were likely to hinder the formation of such solitons. As intermodal group delays are relatively small for a GRIN fiber, it was natural to consider soliton formation in such a medium. Theoretical work carried out during the 1990s indicated that the formation of temporal solitons was indeed feasible inside a GRIN medium [28–30]. It eventually led in 2013 to the observation of a multimode soliton [11]. In this experiment, 300-fs pulses at a wavelength near 1550 nm were launched into a 100-m-long GRIN fiber. However, the spatial width of the beam was so small that only three lowest-order modes of the fiber were excited. The question remained open whether a multimode soliton involving tens or hundreds of modes can form inside a GRIN fiber under suitable conditions.

To answer this question, we look for a pulsed-beam solution that has soliton-like features in the time domain but whose spatial shape evolves in a periodic fashion along the fiber's length. As we discussed in Section 5, Eq. (31) governs just such a solution. It is clear from the presence of f(z) in this modified NLS equation that perfect solitons cannot form inside GRIN fibers. More precisely, this equation is not integrable by the inverse scattering method, which rules out the formation of ideal solitons. However, an equation similar to Eq. (31) was analyzed in 1990 in the context of single-mode fiber links, employing amplifiers periodically for compensating fiber losses. It was found that a new kind of soliton could form inside such fiber links under suitable conditions [31]. We used the same approach in 2018 to show that GRIN fibers support propagation of pulsed Gaussian beams that preserve their temporal shape and behave like a soliton, even though their spatial

width oscillates along the fiber length [16]. We refer to such pulses as GRIN solitons to emphasize that a parabolic index profile is essential for their existence.

Before solving Eq. (31) approximately, it is useful to normalize it in soliton units using the variables [18]

$$\tau = T/T_0, \quad \xi = z/L_D, \quad U = A/\sqrt{P_0},$$
(39)

where  $T_0$  and  $P_0$  are the width and the peak power of input pulses and  $L_D = T_0^2/|\beta_2|$  is the dispersion length. The normalized NLS equation takes the form

$$i\frac{\partial U}{\partial \xi} + \frac{1}{2}\frac{\partial^2 U}{\partial \tau^2} + \frac{N^2}{f(\xi)} \left| U \right|^2 U = 0,$$
(40)

where we assumed  $\beta_2 < 0$  and introduced the soliton order as  $N = (\gamma P_0 L_D)^{1/2}$ . The periodically varying function f(z) given in Eq. (32) can be written as

$$f(\xi) = \cos^2(\pi q\xi) + C^2 \sin^2(\pi q\xi), \qquad q = L_D / z_p,$$
(41)

where  $z_p$  is the self-imaging period introduced in Eq. (23). As we discussed in Section 4, the numerical value of  $z_p$  is < 1 mm for typical GRIN fibers. In contrast, the dispersion length  $L_D$  exceeds 1 m for  $T_0 > 0.1$  ps if we use  $\beta_2 = -20 \text{ ps}^2/\text{km}$ , a typical value near 1550 nm for silica fibers. As a result, q is a large number (q > 100) under typical experimental conditions. Physically, it represents the number of times the spatial beam width oscillates inside a GRIN fiber within one dispersion length.

The dispersion length provides the scale over which solitons evolve [31]. Indeed, solitons cannot respond to beam width changes taking place on a scale of 1 mm or less when  $L_D$  exceeds 1 cm. If we write the solution of Eq. (40) as  $U = \overline{U} + u$ , where  $\overline{U}$  is averaged over one spatial period  $z_p$ , the perturbations  $|u(\xi, \tau)|$  induced by spatial beam-width variations remain small enough that they can be neglected (as long as  $q \gg 1$ ). In other words, the average dynamics of the soliton can be captured by solving the standard NLS equation

$$i\frac{\partial\overline{U}}{\partial\xi} + \frac{1}{2}\frac{\partial^{2}\overline{U}}{\partial\tau^{2}} + \overline{N}^{2}\left|\overline{U}\right|^{2}\overline{U} = 0,$$
(42)

where  $\overline{N}$  is the effective soliton order defined as

$$\overline{N}^2 = N^2 \langle f^{-1}(\xi) \rangle = N^2 c_0 = N^2 / C.$$
(43)

Recall that *C* represents the fraction by which the beam width shrinks at  $z_p/2$  before recovering its original width at  $z = z_p$ . Clearly, Eq. (42) has a solution in the form of a fundamental soliton when we choose  $\overline{N} = 1$  or  $N = \sqrt{C}$ . This solution exists for a wide range of pulse widths  $T_0$  as long as the peak power is adjusted to satisfy this soliton condition. Because of an oscillating beam width, the input peak  $P_0$  must be adjusted to make sure that  $\overline{N} = 1$  on average along the GRIN fiber.

Fig. 4 shows the temporal and spectral profiles of the fundamental



**Fig. 4.** (a) Temporal and (b) spectral profiles of the fundamental GRIN soliton  $(\overline{N} = 1)$  at a distance of  $100L_D$  inside a GRIN fiber with q = 100 and C = 0.45. The corresponding profiles at the input end are shown by dashed lines.



**Fig. 5.** (a) Temporal and (b) spectral profiles of the fundamental GRIN soliton ( $\overline{N} = 1$ ) at a distance of  $10L_D$  inside a GRIN fiber with q = 1 and C = 0.45. The corresponding profiles at the input end are shown by dashed lines.

GRIN soliton ( $\overline{N} = 1$ ) after it has propagated a distance of 100 dispersion lengths inside a GRIN fiber using q = 100 and C = 0.45. These results were obtained by solving Eq. (40) numerically with the initial field  $U(0, \tau) = U_0 \operatorname{sech}(\tau)$ , where  $U_0$  was chosen to ensure  $\overline{N} = 1$ . For comparison, the corresponding profiles at the input end of the fiber are shown as dashed lines. On the logarithmic scale used for the figure, perturbations induced by spatial oscillations can be seen, but their magnitude remains below the 50 dB level even after 100 dispersion lengths. Perturbations become larger as q becomes smaller but remains acceptable even for q = 10. As a worst-case scenario, we consider the q = 1 case for which the dispersion length becomes equal to the selfimaging period ( $L_d = z_p$ ). Fig. 5 compares the input and output profiles (both temporal and spectral) of the fundamental GRIN soliton at a shorter distance of 10L<sub>D</sub>. Remarkably, even in this case, the temporal profile of thee GRIN soliton remains nearly intact after a distances of  $10L_D$ , although its spectrum develops multiple sidebands because of much larger perturbations introduced by periodic self-imaging of the Gaussian beam.

# 8. Soliton fission and dispersive waves

Fission of higher-order solitons is a well-known process that is behind the supercontinuum generation and dispersive-wave emission occurring in single-mode fibers [18]. In this section we consider how this process is affected by the spatial oscillations of a pulsed Gaussian beam related to self-imaging. For this purpose, we modify Eq. (40) to include the effects of third-order dispersion (TOD) and intrapulse Raman scattering:

$$\frac{\partial U}{\partial \xi} + \frac{1}{2} \frac{\partial^2 U}{\partial \tau^2} - i\delta_3 \frac{\partial^3 U}{\partial \tau^3} + \frac{N^2}{f(\xi)} U\left(\xi, \tau\right) \int_0^\infty R(s) \left| U\left(\xi, \tau - s\right) \right|^2 ds = 0,$$
(44)

where  $\delta_3 = \beta_3/(6|\beta_2|T_0)$  is the normalized TOD parameter and the nonlinear response function  $R(t) = (1 - f_R)\delta(t) + f_Rh_R(t)$  includes both the Kerr and Raman contributions with  $f_R = 0.18$ . The form of the Raman response function is given in Ref. [32]. An equation with a periodic nonlinear term similar to Eq. (44) was first solved in Ref. [33]; it was found to produce multiple dispersive waves at different frequencies that agreed well with the experimental results.

We solved Eq. (44) numerically in the frequency domain using the fourth-order Runge-Kutta scheme. Fig. 6 shows the temporal and spectral evolution of a third-order soliton (N = 3) inside a GRIN fiber over one dispersion length using, and  $\delta_3 = 0.02$ . The input pulse is taken to be 100-fs wide ( $T_0 = 57$  fs). The function  $f(\xi)$  is calculated using q = 100 with C = 0.5, a value for which beam width is reduced by a factor of two during each self-imaging period. For comparison, the case of single-mode fibers is shown in Fig. 7 using the same parameters except that C = 1 so that  $f(\xi) = 1$  (no spatial oscillations) in Eq. (44). A direct comparison shows that the soliton dynamics is much richer in the



**Fig. 6.** Temporal and spectral evolution of a third-order soliton (N = 3) inside a GRIN fiber over one dispersion length using  $\delta_3 = 0.02$ , q = 100 and C = 0.5.



**Fig. 7.** Same as Fig. 6 except that the evolution occurs inside a step-index fiber with a constant spatial width such that  $f(\xi) = 1$  (no self-imaging).

case of a GRIN fiber. In both cases, soliton fission occurs and a dispersive wave is produced at a blue-shifted frequency near  $(\nu - \nu_0)T_0 = 5.5$ . However, the fission occurs at a shorter distance in the case of a GRIN fiber, and multiple dispersive waves are generated at both the red and the blue sides of the original spectrum. The intrapulse Raman scattering leads to a red-shift of the shortest soliton in both case, but the shift is much larger in the case of a GRIN fiber.

There are two reasons for the differences seen in Figs. 6 and 7. First, because of the periodic beam compression inside a GRIN fiber, the effective value of the soliton order *N* is enhanced as indicated in Eq. (44). From the values C = 0.5 and N = 3 used for Fig. 6,  $\overline{N}$  is equal to 4.24, i.e., the evolution in Fig. 6 is closer to that occurring for a fourth-order soliton. This is the reason why, fission occurs sooner and the Raman-induced frequency shift is enhanced. Second, periodic self-imaging creates a nonlinear index grating through the Kerr effect because the refractive index is larger in the regions where the beam width is reduced and the intensity is enhanced locally. This grating creates the multiple dispersive waves seen in Fig. 6, both in the temporal and spectral domains. The frequency shift of a dispersive wave from the central frequency  $\omega_0$  of the input pulse,  $\omega = \omega_d - \omega_0$ , can be calculated using a phase-matching condition and is given by [14,33]

$$\frac{\beta_2}{2}\omega^2 + \frac{\beta_3}{6}\omega^3 - \delta\beta_1\omega = \frac{2\pi m}{z_p} + \frac{\gamma P_1}{2C},$$
(45)

where *m* is an integer (positive or negative) and  $\delta\beta_1$  accounts for any change in the group velocity of the soliton from its initial value.

In the preceding equation,  $P_1$  is the peak power of the shortest soliton formed after the fission process is completed. In the case of a GRIN fiber, this power is related to the input peak power  $P_0$  as  $P_1 = P_0(2 - 1/\overline{N})^2$ . We can write Eq. (45) in a normalized form using a new variable  $\Omega = \omega T_0$ :

$$2\delta_3\Omega^3 - \Omega^2 - \delta_1\Omega = 4\pi mq + (2\overline{N} - 1)^2, \tag{46}$$

where  $\delta_1 = \delta\beta_1(L_D/T_0)$ . One can estimate  $\delta_1 \approx 4$  from the slope of the shortest soliton's trajectory in Fig. 6. The real roots of the cubic polynomial in Eq. (46) for different values of *m* provide the frequencies of the dispersive waves that agree reasonably well with those in Fig. 6. For example,  $\Omega/(2\pi) = 5.15$  and 7.22 for m = 0 and m = 1 that match the two vertical lines on the right side of the spectrum seen in Fig. 6. The m = 0 dispersive wave forms only on the blue side of the original spectrum for positive values of  $\delta_3$ . This wave does not require the self-imaging-induced grating and is the only one appearing in Fig. 7. In contrast, this nonlinear grating creates additional pairs of sidebands in Fig. 6 (for  $m \neq 0$ ) on both sides of the original spectrum.

As seen in Fig. 6, the shortest fundamental soliton, created after the fission of a third-order soliton, undergoes a much larger Raman-induced frequency shifts (RIFS) inside a GRIN fiber, compared to a single-mode fiber. The results shown in Fig. 6 are for a specific GRIN fiber for which q = 100 and C = 0.5. We briefly discuss how the RIFS enhancement is affected when these parameters are varied. It is easy to deduce that the RIFS does not depend on the precise value of q as long as the selfimaging period of the GRIN fiber is much shorter than the dispersion length, resulting in q > 10. This is not the case for the *C* parameter. In fact, we expect the RIFS to depend considerably on this parameter because the Gaussian beam is compressed more and more during each self-imaging period as C becomes smaller. The results shown in Fig. 8 confirm this expectation [34]. This figure shows the RIFS as a function of *C* for N = 2 and 3, based on the numerical data obtained by isolating the spectrum of the shortest soliton. As seen there, the RIFS increases rapidly as C decreases and is always larger for a GRIN fiber compared to its value at C = 1 for which the beam width does not oscillate.

The enhancement seen in Fig. 8 is a consequence of the spatiotemporal coupling that occurs invariably in the case of GRIN fibers. Periodic spatial contraction of the pulsed Gaussian beam increases the peak power of the soliton in the middle of each self-imaging cycle, thereby enhancing the nonlinear effects in a periodic fashion. Even though the soliton cannot respond to variations occurring at a length scale of 1 mm or less, the effective value of the soliton order increases, resulting in shorter fundamental solitons after its fission. As the Raman gain is larger for a shorter soliton because of its wider spectrum, the rate of RIFS is also larger for a shorter soliton. This is the reason a larger RIFS occurs in the case of a GRIN fiber for the same input value of N.

# 9. Concluding remarks

In this review I have discussed theory behind the periodic self-



**Fig. 8.** RIFS plotted as a function of *C* for a GRIN fiber of length  $3L_D$  when the input pulses form second (N = 2) and third-order (N = 3) solitons.

imaging of optical beams occurring inside a GRIN fiber, designed such that the refractive index inside the fiber's core decreases in a parabolic fashion from its peak value at the core's center. The self-imaging phenomenon was studied extensively during the decade of the 1970s and was exploited to commercialize the so-called GRIN lens. Even though an input beam incident on a GRIN fiber may excite hundreds of modes, the optical field at any point inside the fiber can be written, without any reference to the fiber modes, as a two-dimensional integral over the input field using a propagation kernel that is similar to that found in diffraction theory. However, this kernel has a specific property that reproduces the input field precisely in a periodic fashion along the length of a GRIN fiber (self-imaging). The physical origin of self-imaging lies in a ladder-like structure of the modal propagation constants with equal spacing between any two neighboring modes of the fiber.

It has been found in recent years that the periodic self-imaging also affects the nonlinear propagation of optical pulses inside multimode GRIN fibers. I applied the general theory of self-imaging to the propagation of a CW Gaussian beam and discussed how self-imaging is modified by self-focusing produced by the Kerr nonlinearity. The case of a pulsed Gaussian beam was studied by following the approach of Ref. [14]. It resulted in a modified NLS equation that includes the effects of periodic spatial beam-width oscillations through a periodically varying effective beam area. I used this equation to discuss the phenomenon of modulation instability inside GRIN fibers. In contrast with the case of single-mode fibers, a large number of spectral sideband pairs can form inside GRIN fibers through a new kind of spatio-temporal instability because o a nonlinear index grating created by the spatial self-imaging phenomenon.

The last two sections show that self-imaging is also behind the formation of GRIN solitons. Self-imaging also produces novel temporal and spectral features when short optical pulses intense enough to form a high-order soliton are launched inside a GRIN fiber. In particular, it leads to considerable enhancement of the RIFS, a feature that is likely to be useful for tuning the wavelength of mode-locked lasers. The polarization effects have been ignored in this review. It is known that self-imaging can lead to considerable changes in the degree of polarization of partially coherent light inside a GRIN fiber even when the nonlinear effects are negligible [35]. In the presence of the Kerr nonlinearity, the state of polarization of an initially coherent beam is expected to evolve inside a GRIN fiber both spatially and temporally, as also observed in a 2018 experiment [36].

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#### Appendix A. Supplementary data

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