

Reflection and transmission of electromagnetic waves at a temporal boundary

Yuzhe Xiao,^{1,*} Drew N. Maywar,² and Govind P. Agrawal¹

¹The Institute of Optics, University of Rochester, Rochester, New York 14627, USA

²Electrical, Computer, and Telecom. Engineering Technology, Rochester Institute of Technology, Rochester, New York 14623, USA

*Corresponding author: yuzhe.xiao@rochester.edu

Received November 19, 2013; revised December 19, 2013; accepted December 19, 2013;

posted December 23, 2013 (Doc. ID 201504); published January 27, 2014

We consider propagation of an electromagnetic (EM) wave through a dynamic optical medium whose refractive index varies with time. Specifically, we focus on the reflection and transmission of EM waves from a temporal boundary and clarify the two different physical processes that contribute to them. One process is related to impedance mismatch, while the other results from temporal scaling related to a sudden change in the speed of light at the temporal boundary. Our results show that temporal scaling of the electric field must be considered for light propagation in dynamic media. Numerical solutions of Maxwell's equations are in full agreement with our theory. © 2014 Optical Society of America

OCIS codes: (260.2110) Electromagnetic optics; (120.5700) Reflection; (120.7000) Transmission.

<http://dx.doi.org/10.1364/OL.39.000574>

Propagation of an electromagnetic (EM) wave through a time-varying medium is an old research topic. As early as 1958, this phenomenon was investigated theoretically in the microwave region [1]. In addition to the generation of a backward-propagating wave, it was predicted that the frequency of the incident wave shifts linearly with changes in the refractive index [1–8]. Several experiments have confirmed such frequency shifts [9–11]. Recently, it was discovered that frequency shifts also occur in the optical regime when light is transmitted through dynamic optical structures in the form of photonic crystals or microring resonators [12–16]. The term “adiabatic wavelength conversion” is often used for such a frequency shift. In the context of plasmas, it is also called photon acceleration [17].

Published literature contains two different sets of expressions for the reflection and transmission coefficients of EM waves at a temporal boundary. One set was derived by Morgenthaler in his 1958 paper using specific boundary conditions [1] and has been used in many other studies since then [3,4,6,9,17,18]. The other set was presented in 1975 using different boundary conditions in the context of plasma [2] and has also been used by others since then [5,8,10,11]. So far, there has been little discussion on the connection between these two different sets of reflection and transmission coefficients.

In this Letter, we revisit this topic and focus on the reflection and transmission of light at a temporal boundary. We show that two different physical processes contribute to them. One process is related to impedance mismatch, while the other results from temporal scaling related to a sudden change in the speed of light at the temporal boundary. The temporal scaling is similar to time dilation of relativity and it leads to “stretching” or “compressing” of temporal slices as light traverses the temporal boundary. Our theory reveals the origin of difference in the sets of reflection and transmission coefficients found in the literature and shows under what conditions they can be used in practice.

Before proceeding, we should clarify the meaning of a “temporal boundary.” The concept of a spatial boundary

between two physical media is quite clear. It is well known that reflection happens when an EM wave strikes a spatial boundary between two optical media with different refractive indexes. In this case, the Fresnel's equations provide expressions for the reflection and transmission coefficients [19]. Now imagine the situation in which refractive index of a medium is uniform in space but changes suddenly everywhere at a specific instant of time. In this case, we have a temporal boundary with a discontinuity of the refractive index on its opposite sides. This Letter focuses on such a temporal boundary. We emphasize that we are not considering a moving medium in which a spatial boundary moves with time.

Figure 1 shows what happens to a plane wave, polarized along the x axis and propagating along the z axis, as it approaches a temporal boundary located at $t = t_0$, where the refractive index changes from n_1 to n_2 . Before the boundary ($t < t_0$), the plane wave is propagating at a speed c/n_1 . At the boundary ($t = t_0$), its speed changes to c/n_2 , and at the same time a reflected wave is created with its electric and magnetic field oriented as shown in Fig. 1. The reflected wave will be generated only if the medium impedance changes at the temporal

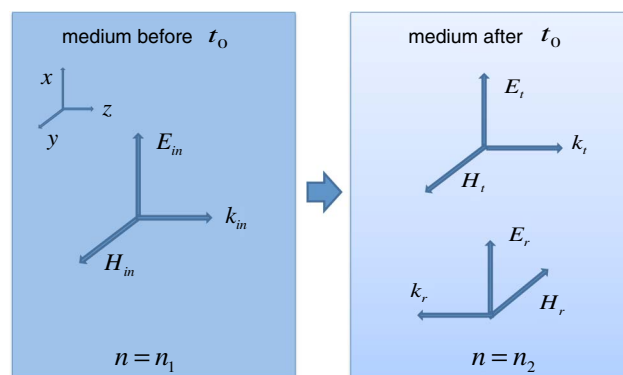


Fig. 1. Schematic illustration of reflection and transmission of an EM fields at a temporal boundary where the refractive index changes from n_1 (left panel) to its final value n_2 (right panel) at $t = t_0$.

boundary, i.e., if $\eta_1 = (\mu_1/\varepsilon_1)^{1/2}$ is different from $\eta_2 = (\mu_2/\varepsilon_2)^{1/2}$, where ε and μ represent the dielectric permittivity and magnetic permeability of the medium, respectively, and the subscripts 1 and 2 denote these quantities before and after the temporal boundary located at $t = t_0$.

In the following discussion, we denote the electric and magnetic fields before $t = t_0$ as \mathbf{E}_{in} and \mathbf{H}_{in} . After the index change, the transmitted and reflected electric fields are denoted by \mathbf{E}_t and \mathbf{E}_r , respectively, with a similar notation for the magnetic field (see Fig. 1). Transmission and reflection at a temporal boundary can be characterized by coefficients similar to Fresnel's equations at a spatial boundary. However, there is considerable confusion in the published literature as two different sets of expressions exist for the transmission and reflection coefficients. We review briefly the origin of this discrepancy.

Both cases start from Maxwell's equations. In the absence of free charges and currents, these are given by

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{H} = -\frac{\partial \mathbf{D}}{\partial t}, \quad (1)$$

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \cdot \mathbf{B} = 0, \quad (2)$$

where the electric displacement \mathbf{D} and magnetic induction \mathbf{B} are related to the EM fields \mathbf{E} and \mathbf{H} as $\mathbf{D} = \varepsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$. The two sets of expressions differ in their choice of the boundary conditions that the four fields must satisfy across the temporal boundary.

One approach argues that the quantities \mathbf{D} and \mathbf{B} should remain continuous across the temporal boundary in view of the presence of their time derivatives in Eq. (1):

$$\mathbf{D}(t = t_0^-) = \mathbf{D}(t = t_0^+), \quad \mathbf{B}(t = t_0^-) = \mathbf{B}(t = t_0^+). \quad (3)$$

These boundary conditions ensure that Maxwell's equations remain valid for all times, including the instant t_0 when the refractive index changes suddenly. In this case, the transmission and reflection coefficients for the electric field are found to be

$$r_{\text{DB}} \equiv \frac{E_r}{E_{\text{in}}} = \frac{1}{2} \left(\frac{\varepsilon_1}{\varepsilon_2} - \frac{\sqrt{\mu_1 \varepsilon_1}}{\sqrt{\mu_2 \varepsilon_2}} \right), \quad (4)$$

$$t_{\text{DB}} \equiv \frac{E_t}{E_{\text{in}}} = \frac{1}{2} \left(\frac{\varepsilon_1}{\varepsilon_2} + \frac{\sqrt{\mu_1 \varepsilon_1}}{\sqrt{\mu_2 \varepsilon_2}} \right). \quad (5)$$

These equations were first derived in [1] and have been used extensively since then [3,4,6,9,17,18].

In the second approach, one insists on the continuity of the electric and magnetic fields, \mathbf{E} and \mathbf{H} , and assumes that they cannot change abruptly at t_0 when the refractive index of the medium changes at a temporal boundary:

$$\mathbf{E}(t = t_0^-) = \mathbf{E}(t = t_0^+), \quad \mathbf{H}(t = t_0^-) = \mathbf{H}(t = t_0^+). \quad (6)$$

This assumption leads to another set of expressions for the reflection and transmission coefficients for the electric field:

$$r_{\text{EH}} \equiv \frac{E_r}{E_{\text{in}}} = \frac{1}{2} \left(1 - \frac{\sqrt{\mu_2 \varepsilon_1}}{\sqrt{\mu_1 \varepsilon_2}} \right), \quad (7)$$

$$t_{\text{EH}} \equiv \frac{E_t}{E_{\text{in}}} = \frac{1}{2} \left(1 + \frac{\sqrt{\mu_2 \varepsilon_1}}{\sqrt{\mu_1 \varepsilon_2}} \right). \quad (8)$$

These results were first obtained in 1975 in the context of plasma [2] and have been used in several works since then [5,8,10,11].

Clearly, only one set of these coefficients can be right as Eqs. (3) and (6) cannot be satisfied simultaneously at a temporal boundary. The question is which one? Here we show that Eq. (3) is the correct choice, as also expected intuitively on the physical ground. To fully understand what happens at a temporal boundary, we need to consider the special case in which the refractive index is changed such that the impedance η does not change across the temporal boundary ($\eta^2 = \mu_1/\varepsilon_1 = \mu_2/\varepsilon_2$), and no light is reflected ($r_{\text{DB}} = 0$). Naively one would think that t_{DB} must be 1 to meet the "expected" 100% of transmission through the temporal boundary. [Indeed $t_{\text{EH}} = 1$, and this might be the reason that Eqs. (7) and (8) were adopted.] However, Eq. (5) predicts $t_{\text{DB}} \neq 1$. To claim that Eq. (3) is the correct choice, we need to understand the physical origin for changes in the amplitude of the transmitted electric field in the case of matched impedances.

A clue is found by noting that the two sets of transmission and reflection coefficients are not independent and can be related as

$$r_{\text{DB}} = -\frac{n_1}{n_2} r_{\text{EH}}, \quad t_{\text{DB}} = \frac{n_1}{n_2} t_{\text{EH}}, \quad (9)$$

where $n_i = \sqrt{\varepsilon_i \mu_i / (\varepsilon_0 \mu_0)}$ with $i = 1$ or 2 . The same ratio n_1/n_2 was also found to occur in our recent work [20] in the context of time-dependent changes in the refractive index of an optical medium.

Motivated by the discovery of adiabatic wavelength conversion in microring resonators and photonic-crystal waveguides [14–16], we recently developed a theory of time transformation in dynamic optical media [20,21]. Our theory deals directly with changes in the electric field occurring when an optical field propagates through a medium whose refractive index changes with time. In the case of a nondispersive linear medium, we obtained the following analytical expression for the transmitted field for an input field $E_{\text{in}}(t)$ [20]:

$$E_t(t) = s E_{\text{in}}(st - sT_d), \quad (10)$$

where T_d is a linear transit delay and $s = n_1/n_2$ is the time-dilation factor related to change in the speed of light after the temporal boundary. As we discussed in [20], time-dependent changes in the refractive index of a linear medium affect the electric field in three distinct ways: (1) a shift in the carrier frequency, (2) a change in the temporal duration of an optical cycle, and (3) a change in the amplitude of the electric field. In the case of an optical pulse, these modifications change the carrier frequency, the temporal duration, and the peak amplitude at

the same time. Since our work assumed impedance matching implicitly (reflection was ignored) and leads to a correct change in the transmitted-field amplitude, it is consistent with the boundary condition Eq. (3).

The physical origin of the time-dilation factor s is related to the time transformation [21] that maps the input field to the output field. More precisely, each temporal slice of the electric field is either “stretched” or “compressed” depending on whether the speed of light decreases or increases after passing the temporal boundary. Equation (10) shows that such time dilation is accompanied with a linear scaling of the field amplitude by the factor s . More importantly, the amplitude modification factor s is the factor relating two sets of transmission and reflection coefficients in Eq. (9). Thus, we conclude that the transmission coefficient in Eq. (5) is the product of two factors that have different physical origins. Indeed, it can be written as $t_{DB} = st_{EH}$. The factor s is due to the time dilation, whereas the factor t_{EH} has its origin in the impedance mismatch across a temporal boundary. The use of “wrong” boundary conditions in Eq. (6) misses completely the factor s . Clearly, the boundary conditions in Eq. (3) must be used since they capture both physical effects. The minus sign appearing in Eq. (9) can be understood from our time-transformation theory by noting each slice of the reflected electric field reverses its propagation direction upon reflection; this time reversal results in the factor -1 through time transformation.

Previous works did point out changes in the optical frequency across a temporal boundary, which is one of the consequences of time dilation, but most of them focused on the CW case. Therefore, “stretching” or “compression” of the electric field was not discussed. Although numerical simulations in [18] showed changes in the width of an optical pulse in dynamic media, the physical reason behind it was not discussed in that work. Here we show that time dilation should not be ignored, as it modifies the amplitudes for the transmitted electric by a factor of s . This factor is also needed to ensure the validity of the boundary conditions given in Eq. (3).

Since Maxwell’s equations contain the whole physics, any solution of these equations should agree with our conclusions. To verify this, we solved them numerically using the well-known finite-different time-domain (FDTD) method. For simplicity, we consider a one-dimensional propagation problem where the electric field is in the form of a four-cycle pulse (with its peak amplitude normalized to 1) and propagates in the z direction as a plane wave (without any diffraction in the transverse plane). We assume that a temporal boundary exists where the medium parameters ϵ and μ change instantaneously.

In case 1, we increase the permittivity of the whole medium uniformly from $\epsilon_1 = 2\epsilon_0$ to $\epsilon_2 = 1.5625\epsilon_1$ but keep the permeability unchanged ($\mu_1 = \mu_2 = \mu_0$). Such a change increases the refractive index by 25% ($s = 0.8$). Since the impedance of the medium also changes, a reflected wave must be generated in this case. The electric field obtained numerically after the temporal boundary is plotted in Fig. 2. The peak amplitude of the transmitted field is 0.72. One can also clearly see that a part of the pulse energy is reflected back. The peak

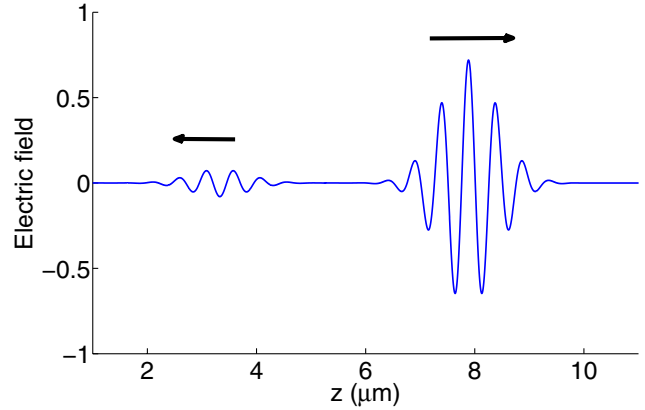


Fig. 2. Electric field of a four-cycle pulse after propagation across a temporal boundary. Reflected wave is generated because of impedance mismatch. The amplitudes of the reflected and transmitted fields agree with Eqs. (4) and (5).

amplitude of the reflected field is 0.08, and its phase is shifted by π . Using the parameter values employed in numerical simulations, we find that Eqs. (4) and (5) give the correct amplitude for both the reflected and transmitted fields.

In case 2, we change both the permittivity and permeability of the medium in such a way that the impedance of the medium does not change at the temporal boundary even though the refractive index does change. More specifically, we choose $\epsilon_2 = 1.25\epsilon_1$ and $\mu_2 = 1.25\mu_0 = 1.25\mu_1$ so that the refractive index of the medium changes again by 25%. Figure 3 shows the electric field obtained numerically in this situation. As one might have expected, FDTD simulations confirm that no reflections occur in the presence of impedance matching. The amplitude of the transmitted field is still reduced in this case, with a peak value of 0.8, which agrees with the prediction of Eq. (5). Clearly, this reduction is not related to energy transfer to a reflected wave because no reflection occurs. As we discussed earlier, this reduction is related to the prediction of time transformation theory in Eq. (10). Indeed, one can immediately verify that $s = n_1/n_2 = 0.8$, in agreement with the numerically obtained value of 0.8 in Fig. 3. Amplitude changes of

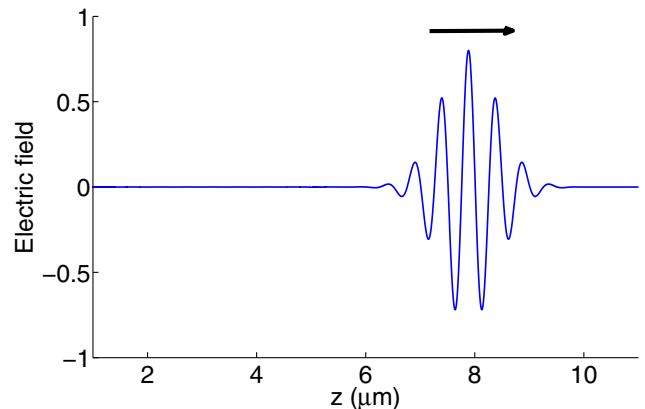


Fig. 3. Electric field of a four-cycle pulse after propagation across a temporal boundary with impedance matching. No electric field is reflected back. The amplitudes of the reflected and transmitted fields agree with Eqs. (4) and (5).

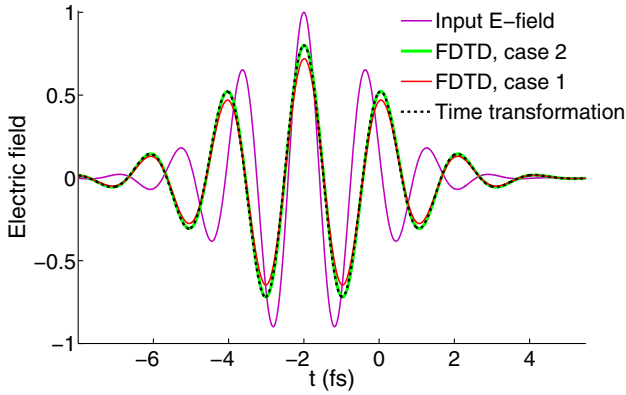


Fig. 4. Comparison of transmitted electric fields in cases 1 (red curve, impedance changes) and 2 (green curve, impedance matched). Dotted black curve shows the analytic prediction based on Eq. (10). The input electric field is shown for comparison with a purple curve. FDTD simulations confirm our prediction and agree fully with our time transformation theory.

transmitted light due to stretching or compressing of temporal slices, to the best of our knowledge, was first pointed out by us in [20]. (For a discussion of energy conservation under such conditions, we refer to [14]).

To take a closer look at how the transmitted field changes in the preceding two cases, we plot in Fig. 4 the transmitted electric fields in case 1 (red curve) and case 2 (green curve) on top of the input electric field (purple curve). The prediction of time-transformation theory [Eq. (10)] in case 2, shown with a dotted black curve, is in excellent agreement with FDTD simulations. More specifically, the amplitude of the field is modified by a factor of 0.8; the carrier frequency is also modified by the same factor with the corresponding change in the duration of an optical cycle. When the impedance is not matched across the temporal boundary (case 1), the transmitted field looks almost the same as in case 2. The only difference is that its amplitude is further reduced from 0.8 to 0.72. This reduction is solely due to reflection and its magnitude is given by $t_{EH} = 0.9$ given in Eq. (8). The FDTD simulations shown in Figs. 2–4 confirm the conclusion reached earlier based on the time-transformation method that Eqs. (3)–(5) provide correct description of reflection and transmission of an EM field at a temporal boundary.

In conclusion, this Letter focuses on the reflection and transmission of EM waves at a temporal boundary and resolves a discrepancy that have existed in literature for more than 40 years. We show that the results based on the continuity of \mathbf{D} and \mathbf{B} across a temporal boundary provide the correct physical description. Using our recently developed time-transformation theory, we were

able to show that two different physical processes contribute to the transmitted field. One of them is related to the impedance mismatch. The other one has its origin in time dilation occurring because of a sudden change in the speed of light. Our time-transformation theory provides an analytic expression for the transmitted field under such conditions. It can be applied to pulses of any durations (as short as a single cycle) and shows that the transmitted pulse undergoes not only an amplitude change but also experiences a change in its width together with a shift in its carrier frequency. Numerical solutions of Maxwell's equations are in full agreement with our theory. Our results provide physical insight and should be useful in many diverse areas of physics involving propagation of EM radiation at frequencies ranging from microwaves to x rays.

References

1. F. R. Morgenthaler, IRE Trans. Microwave Theory Tech. **6**, 167 (1958).
2. C.-L. Jiang, IEEE Trans. Antennas Propag. **23**, 83 (1975).
3. T. Ruiz, C. Wright, and J. Smith, IEEE Trans. Antennas Propag. **26**, 358 (1978).
4. J. C. AuYeung, Opt. Lett. **8**, 148 (1983).
5. D. K. Kalluri and V. R. Goteti, J. Appl. Phys. **72**, 4575 (1992).
6. M. Cirone, K. Rzaz-dotewski, and J. Mostowski, Phys. Rev. A **55**, 62 (1997).
7. J. T. Mendonca and P. K. Shukla, Phys. Scr. **65**, 160 (2002).
8. D. K. Kalluri and C. Jinning, IEEE Trans. Antennas Propag. **57**, 2698 (2009).
9. S. C. Wilks, J. M. Dawson, and W. B. Mori, Phys. Rev. Lett. **61**, 337 (1988).
10. N. Yugami, T. Niiyama, T. Higashiguchi, H. Gao, S. Sasaki, H. Ito, and Y. Nishida, Phys. Rev. E **65**, 036505 (2002).
11. A. Nishida, N. Yugami, T. Higashiguchi, T. Otsuka, F. Suzuki, M. Nakata, Y. Sentoku, and R. Kodama, Appl. Phys. Lett. **101**, 161118 (2012).
12. M. F. Yanik and S. Fan, Phys. Rev. Lett. **92**, 083901 (2004).
13. M. F. Yanik and S. Fan, Phys. Rev. Lett. **93**, 173903 (2004).
14. M. Notomi and S. Mitsugi, Phys. Rev. A **73**, 051803(R) (2006).
15. S. Preble, Q. Xu, and M. Lipson, Nat. Photonics **1**, 293 (2007).
16. T. Kampfrath, D. M. Beggs, T. P. White, A. Melloni, T. F. Krauss, and L. Kuipers, Phys. Rev. A **81**, 043837 (2010).
17. J. T. Mendonca, *Theory of Photon Acceleration* (Taylor & Francis, 2000).
18. F. Biancalana, A. Amann, A. V. Uskov, and E. P. O'Reilly, Phys. Rev. E **75**, 046607 (2007).
19. J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, 1998), p. 302.
20. Y. Xiao, G. P. Agrawal, and D. N. Maywar, Opt. Lett. **36**, 505 (2011).
21. Y. Xiao, D. N. Maywar, and G. P. Agrawal, Opt. Lett. **37**, 1271 (2012).