Propagation of few-cycle pulses in nonlinear Kerr media: harmonic generation

Yuzhe Xiao,[1,*] Drew N. Maywar,[2] and Govind P. Agrawal[1]

[1]*The Institute of Optics, University of Rochester, Rochester, New York 14627, USA*

[2]*Electrical, Computer, and Telecommunications Engineering Technology, Rochester Institute of Technology, Rochester, New York 14623, USA*

**Corresponding author: yuxiao@optics.rochester.edu*

Received December 27, 2012; accepted January 18, 2013;
posted January 29, 2013 (Doc. ID 182431); published February 26, 2013

We apply our recently developed time-transformation method for studying the propagation of few-cycle optical pulses inside a nonlinear Kerr medium after taking into account that changes in the refractive index vary with the electric field as E^2 and not by its average over an optical cycle. Our technique correctly predicts carrier-wave shocking and generation of odd-order harmonics inside a Kerr medium, the two features found earlier with directly solving Maxwell's equations using the finite-difference time-domain (FDTD) methods. We extend our method to study the impact of a finite response of the Kerr nonlinearity on harmonic generation and to include chromatic dispersion that cannot be ignored for ultrashort pulses. We show that nonlinear effects can help in controlling the width of an ultrashort pulse, even though it cannot propagate as a fundamental soliton. Our time-transformation method provides an alternative to the FDTD technique, as it deals with the electric field directly but does not require step size to be a small fraction of the wavelength, resulting in much faster computation speeds.

OCIS codes: 190.5530, 320.7110.

The development of ultrafast optical technology enables the generation of ultrashort pulses containing only a few cycles of the electromagnetic field. Many applications require propagation behavior of such short pulses inside a dispersive nonlinear Kerr medium. Although the nonlinear Schrödinger equation, derived from Maxwell's equations using the slowly varying envelope approximation [1], cannot be used for such short pulses, a generalized version of this equation has been used with success for pulses as short as single cycle [2]. However, in this envelope-based approach, the Kerr nonlinearity is included using the form $n = n_0 + n_2\langle E^2(t)\rangle$ for the refractive index, where n_2 is the Kerr parameter and the average is over a few optical cycles. Although it is possible to extend the envelope approach further by adding additional nonlinear terms [3], it is common to employ the finite-difference time-domain (FDTD) method and solve Maxwell's equations directly in the time domain [4]. The FDTD approach shows that the use of $E^2(t)$ in place of its average leads to carrier-wave shocking and generation of odd-order harmonics when a pulse propagates inside a Kerr medium [5]. However, the use of FDTD algorithm requires a step size that is a small fraction of wavelength λ. As a result, its use becomes time consuming for distances much longer than λ.

We recently developed a time-transformation technique to propagate the pulse through a Kerr medium by performing a nonlinear mapping of the electric field. This approach does not require step sizes much shorter than λ [6]. Chromatic dispersion can be easily included into this approach [7]. However, in our previous work, we used $\langle E^2\rangle$ in including the Kerr nonlinearity. Here we apply the time-transformation method to study propagation of few-cycle pulses with $n = n_0 + n_2E^2(t)$. Our approach correctly predicts the formation of carrier-wave shocking and generation of harmonics and agrees fully with the FDTD approach. We then extend our analysis by including both the chromatic dispersion and a finite response of the Kerr nonlinearity.

In the time-transformation method, propagation through a nonlinear medium of length L is governed by [7]

$$E_{\mathrm{out}}(t) = \int_{-\infty}^{\infty} h(t - t' - T_d)E_{\mathrm{in}}(t')\mathrm{d}t', \tag{1}$$

where $T_d(t')$ represents the delay of a temporal slice located at t'. We convert this equation into a convolution

$$E_{\mathrm{out}}(t) = \int_{-\infty}^{\infty} h(t - t_1)E'(t_1)J(t_1)\mathrm{d}t_1, \tag{2}$$

by the nonlinear temporal mapping $t_1 = t' + T_d(t')$. Here $h(t)$ is the Fourier transform of the frequency response function $H(\omega) = \exp[in_0(\omega)\omega L/c]$ and the Jacobian of the time transformation is given by $J(t_1) = dt'/dt_1$.

The time delay has two parts, $T_d = T_l + T_{\mathrm{nl}}$. The linear part, $T_l = n_0L/c$, comes from the linear part of the refractive index, and the nonlinear part for a Kerr medium is given by

$$T_{\mathrm{nl}}(t) = \frac{n_2L}{c}\int_{-\infty}^{t} R(t - t')E^2(t')\mathrm{d}t', \tag{3}$$

where $R(t)$ is the third-order nonlinear response function of the medium. It is important to stress that the length L in Eq. (3) should be a small fraction of the nonlinear length [1]. In practice, the nonlinear medium is divided into multiple sections, and Eq. (2) is used repetitively to propagate the pulse through each section. Because of the use of E^2 in Eq. (3), instead of its cycle-average value, T_{nl} oscillates at a frequency twice that of the input field. As will be seen later, these rapid oscillations generate odd-order harmonics within the Kerr medium.

As a simple test of the validity of our time-transformation method, we first consider a nondispersive Kerr medium. We assume that the Kerr medium responds instantaneously and use $R(t - t') = \delta(t - t')$ in Eq. (3). The

0146-9592/13/050724-03$15.00/0

input field oscillating at the frequency $f_c = 200$ THz has a Gaussian envelope with a width parameter $T_0 = 5$ fs:

$$E_{\rm in}(t) = E_0 \exp[-t^2/(2T_0^2)] \cos(2\pi f_c t). \quad (4)$$

The peak amplitude E_0 of the electric field is set by the maximum nonlinear phase shift it experiences, $\phi_{\max} = 2\pi f_c n_2 E_0^2 L = 0.3$.

Figure 1 shows the electric fields and corresponding optical spectra at the input and output ends of the Kerr medium using our approach and compares them with those obtained with the FDTD method. The agreement between the two is excellent. The output electric field is distorted because of carrier-wave shocking [5], which happens mainly near the pulse center, where the Kerr effect is the strongest. The distortion of the electric field is due to the generation of odd-order harmonics that are seen clearly in Fig. 1(b) up to $9f_c$. Physically speaking, a Kerr medium modulates the refractive index in a periodic fashion at $2f_c$, creating a moving index grating, which creates spectral sidebands seen in Fig. 1(b). Mathematically, the change in the refractive index can be expressed as

$$\Delta n = \frac{1}{2} n_2 E_0^2 \exp(-t^2/T_0^2)[1 + \cos(4\pi f_c t)]. \quad (5)$$

The first term (without f_c) is responsible for self-phase modulation, while the second term generates odd-order harmonics.

The instantaneous response of a Kerr medium is clearly an approximation because any electronic response should be delayed by some finite time. While this approximation is justified for relatively wide pulses, it becomes questionable for ultrashort pulses. To relax it, we introduce a finite response time τ_k for the Kerr nonlinearity through a commonly used Debye model [8]. It assumes the following exponential form of the response function $R(t)$ in Eq. (3):

$$R(t) = \tau_k^{-1} \exp(-t/\tau_k). \quad (6)$$

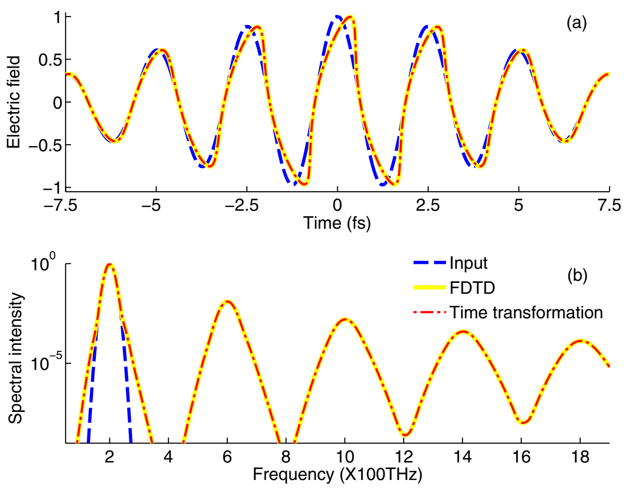

Fig. 1. (Color online) (a) Electric fields and (b) optical spectra at the input (dashed blue curves) and output ends of a nondispersive Kerr medium using our new approach (dot-dashed red curves) and the FDTD method (solid yellow curves).

In the limit $\tau_k = 0$, $R(t)$ is reduced to a δ function, as expected. We performed a series of simulations where we reduced τ_k from 2 to 0.05 fs, while keeping all other parameters the same as in Fig. 1. Figure 2 shows how the peak amplitudes of the third, fifth, and seventh harmonics vary with τ_k. As seen there, the amplitudes of all harmonics decrease almost exponentially as τ_k increases (notice the semi-log nature of the plot), but the rate of decrease is different for different harmonics. The third harmonic decreases the least [8], and the rate becomes larger for higher-order harmonics. This behavior can be understood by noting that the Δn in Eq. (5) oscillates at a frequency of 400 THz. In order to resolve these dynamics, the response time should be well below 2.5 fs.

We now consider the case of a dispersive Kerr medium since dispersion cannot be ignored for ultrashort pulses. As is well known, optical solitons can form in a dispersive Kerr medium with a delicate balance between the nonlinear and dispersive effects and when pulses with a specific shape and peak intensity are launched. It is not expected that this balance will persist for few-cycle pulses that are effected by higher-order effects, such as self-steepening and higher-order dispersion. Nonetheless, it is possible that pulses will remain close to their original shape. To see this effect numerically, we consider an optical pulse with a sech-shape envelope and replace the Gaussian factor in Eq. (4) with $\mathrm{sech}(t/T_0)$ and use $T_0 = 10$ fs. In order to include higher-order dispersive effects for such ultrashort pulses, we employ the Lorentz model for the dielectric constant of the Kerr medium in the form

$$\varepsilon(\omega) = \varepsilon_\infty + \frac{\omega_0^2(\varepsilon_s - \varepsilon_\infty)}{\omega_0^2 - i\delta\omega - \omega^2}. \quad (7)$$

The four parameters appearing in this model are chosen as $\varepsilon_s = 5.25$, $\varepsilon_\infty = 2.25$, $\omega_0 = 6 \times 10^{14}$ rad/s, and $\delta = 2 \times 10^9$ s^{-1}. The second-order dispersion parameter β_2 of such a Kerr medium varies from −4 to −20 ps^2/m over the frequency range 180–220 THz, a range that includes $f_c = 200$ THz used in our simulations. The peak

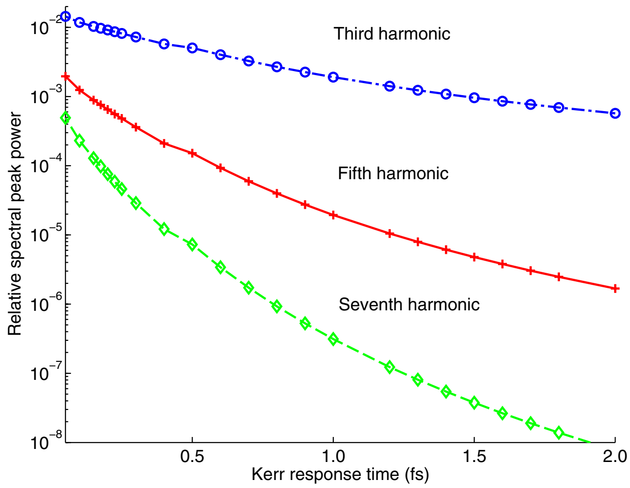

Fig. 2. (Color online) Changes in the relative amplitudes of the first three harmonics with the Kerr response time τ_k. In all cases, the amplitude decreases almost exponentially with increasing τ_k.

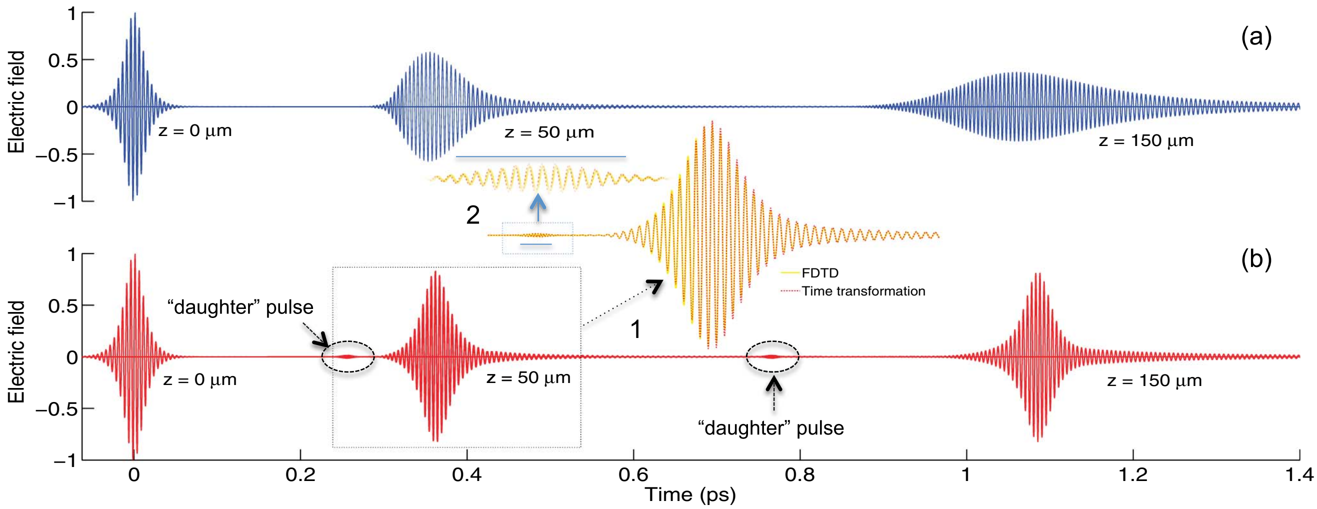

Fig. 3. (Color online) Electric fields of a 10 fs optical pulse after it has propagated for 50 μm ($4L_D$) and 150 μm ($12L_D$) in a (a) linear and (b) nonlinear dispersive medium. (Inset) Comparison of the time-transformation (dashed red) and the FDTD (solid yellow) methods in the nonlinear case; an expanded view of the daughter pulse is also shown.

amplitude E_0 in Eq. (4) is chosen to satisfy the soliton-formation condition [1] and is found using $n_2 E_0^2 = 0.0182$.

To see whether soliton-like features persist, we propagate the few-cycle pulse in such a dispersive Kerr medium for distance up to 150 μm and compare the results with the case when nonlinearity is removed by setting $n_2 = 0$. Figure 3 shows the electric field of the pulse in these two cases at distances of 50 and 150 μm, corresponding to $4L_D$ and $12L_D$, respectively, where $L_D = T_0^2/|\beta_2|$ is the dispersion length [1]. In a dispersive medium without any nonlinear effects ($n_2 = 0$), pulse broadens rapidly as expected [Fig. 3(a)]. However, when Kerr nonlinearity is include, pulse width remains close to its original shape, indicating soliton-like propagation [Fig. 3(b)]. It is also clear from Fig. 3 that the pulse does not maintain its original shape fully and develops a long tail whose origin is related to the presence of higher-order dispersive effects and self-steepening. A weak "daughter" pulse that oscillates at a much higher frequency can also be seen in Fig. 3. The origin of this daughter pulse lies in the odd-order harmonics generated by the Kerr nonlinearity (see Fig. 1). In the nondispersive case, all frequency components overlap temporally, leading to carrier-wave shocking. In a dispersive medium, because of a frequency-dependent group velocity, various harmonics travel at different speeds and form a daughter pulse, which separates from the main pulse. This separation becomes larger at longer distances, as seen in Fig. 3.

As a further check of the accuracy of our results, we solved the same propagation problem with the FDTD method. The central inset in Fig. 3 compares the FDTD results at a distance of 50 μm with those obtained by the time-transformation method. Both the amplitudes and phases of the transmitted electric field agree quite well. However, the time-transformation method has a distinct advantage over the FDTD technique in terms of the computation speed. In our tests (using MATLAB, version 7.8), our method was more than 10 times faster than the FDTD method for attaining the same accuracy for the results shown in the inset of Fig. 3. The reason is that the section length in our method is not related to the temporal resolution employed.

In conclusion, we apply our recently developed time-transformation method for studying the propagation of few-cycle optical pulses inside a nonlinear Kerr medium with $n = n_0 + n_2 E^2(t)$. Our technique correctly predicts carrier-wave shocking and generation of odd-order harmonics, the two features found earlier with the FDTD method. The time-transformation method makes it relatively easy to take into account a finite nonlinear response time. We used this feature to study the impact of a finite response time of the Kerr nonlinearity on harmonic generation. As a further extension, we include chromatic dispersion using a Lorentz model so that dispersion to all orders can be included for few-cycle pulses. We show numerically that nonlinear effects can help in controlling the width of an ultrashort pulse, even though it cannot propagate as a fundamental soliton. Our time-transformation method provides an alternative to the FDTD technique as it deals with the electric field directly but does not require step size to be a small fraction of the wavelength, resulting in much faster computation speeds. It should prove quite useful in the fields of nonlinear optics, ultrafast optics, and terahertz optics dealing with single or few-cycle pulses.

References

1. G. P. Agrawal, *Nonlinear Fiber Optics*, 5th ed. (Academic, 2012).
2. T. Brabec and F. Krausz, Phys. Rev. Lett. **78**, 3282 (1997).
3. G. Genty, P. Kinsler, B. Kibler, and J. M. Dudley, Opt. Express **15**, 5382 (2007).
4. P. M. Goorjian and A. Taflove, Opt. Lett. **17**, 180 (1992).
5. R. G. Flesch, A. Pushkarev, and J. V. Moloney, Phys. Rev. Lett. **76**, 2488 (1996).
6. Y. Xiao, G. P. Agrawal, and D. N. Maywar, Opt. Lett. **37**, 1271 (2012).
7. Y. Xiao, D. N. Maywar, and G. P. Agrawal, J. Opt. Soc. Am. B **29**, 2958 (2012).
8. L. Gilles, J. V. Moloney, and L. Vázquez, Phys. Rev. E **60**, 1051 (1999).