New approach to pulse propagation in nonlinear dispersive optical media

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We develop an intuitive approach for studying propagation of optical pulses through nonlinear dispersive media. Our new approach is based on the impulse response of linear systems, but we extend the impulse response function using a self-consistent time-transformation approach so that it can be applied to nonlinear media as well. Numerical calculations based on our new approach show excellent agreement with the generalized nonlinear Schrödinger equation in the specific case of the Kerr nonlinearity in both the normal and anomalous dispersion regimes. An important feature of our approach is that it works directly with the electric field associated with an optical pulse and can be applied to pulses of arbitrary width. Numerical calculations performed using single-cycle optical pulses show that our results agree with those obtained with the finite-difference time-domain technique using considerably more computing resources.

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1. INTRODUCTION

Propagation of optical pulses through nonlinear dispersive media is studied in many contexts ranging from optical communications [1] to ultrafast optics [2]. The usual starting point is the nonlinear Schrödinger (NLS) equation, obtained from Maxwell's equations under the slowly varying envelope approximation (SVEA), and its predictions are quite accurate as long as pulses are not so short that the SVEA begins to break down [3]. Attempt has been made to modify the NLS equation so that it could be applied to pulses as short as a single optical cycle [4]. In recent years, attosecond pulses are attracting considerable attention for a variety of applications [5,6]. Numerical methods that integrate Maxwell's equations directly, such as the finite-difference time-domain (FDTD) method, are widely used for ultrashort optical pulses [7–9]. Although the FDTD method solves the pulse propagation problem with the least approximations, it is relatively time-consuming and is typically useful only for propagation distances shorter than 1 mm.

In this paper, we propose a novel intuitive approach for studying propagation of optical pulses through a nonlinear dispersive medium. It makes use of the impulse response function associated with a linear system but extends it so that it can be applied to a nonlinear medium as well. As our approach works directly with the electric field associated with an optical pulse, it does not have any requirement on the pulse width and could be used for ultrashort pulses containing a single optical cycle, a region where the NLS equation is likely to fail. The basic theory of linear systems is introduced, and the impulse response function of a linear dispersive medium is obtained using a standard approach in Section 2. This impulse response function is extended to the nonlinear case in Section 3 using a self-consistent physical approach. In particular, a nonlinear medium introduces an additional transit time delay that is different for different parts of the optical pulse (depending on the local intensity of each temporal slice). Numerical examples, discussed in Sections 4 and 5 for long and short pulses, show that this new approach agrees quite well with the generalized NLS equation and the FDTD method in the appropriate limit. The main results are summarized in the final concluding section.

2. PULSE PROPAGATION IN A LINEAR DISPERSIVE MEDIUM

For any linear system, the input and output signals, $x(t)$ and $y(t)$, respectively, are related as

$$y(t) = \int_{-\infty}^{\infty} h(t, t')x(t')\mathrm{d}t', \tag{1}$$

where $h(t, t')$ is the system's impulse response function. For a linear time-invariant system, the impulse response function depends only on one single variable, the time difference, and takes the form $h(t - t')$. In this case, Eq. (1) becomes a convolution and takes a very simple form in the frequency domain:

$$\tilde{y}(\omega) = H(\omega)\tilde{x}(\omega), \tag{2}$$

where $\tilde{y}(\omega)$ and $\tilde{x}(\omega)$ are the Fourier transforms of the output and input signals. The frequency response function $H(\omega)$ is the Fourier transform of $h(t)$. Equation (2) shows that such a linear system acts like a filter in the frequency domain.

Can the linear system theory be applied to propagation of optical pulses through a linear dielectric medium? The answer is not obvious because, in general, one must solve Maxwell's equations for the electric field $\mathbf{E}(\mathbf{r}, t)$, which is a vector quantity that depends both on time t and spatial position $\mathbf{r}$.

The vector nature can be ignored if the polarization properties do not change during propagation. The transverse spatial coordinates can also be ignored when pulses propagate through a single-mode optical waveguide. In the following, we assume that both of these requirements are met and introduce a scalar field using the relation $\mathbf{E}(\mathbf{r}, t) = \hat{e}F(x, y)E(t - z/v_g)$, where $\hat{e}$ is the polarization unit vector, $F(x, y)$ is the spatial distribution of the waveguide mode, and v_g is the group velocity. As illustrated in Fig. 1, a dielectric medium transforms the electric field associated with the optical pulse. If we denote the input and output fields by E_{in} and E_{out}, respectively, Eq. (1) can be written as

$$E_{\mathrm{out}}(t) = \int_{-\infty}^{\infty} h(t - t')E_{\mathrm{in}}(t')\mathrm{d}t'. \tag{3}$$

Once the impulse response function of an optical medium is known, the pulse propagation problem is completely solved.

A dispersive linear medium, whose refractive index changes with frequency, can be thought of as a time-invariant linear system. The impulse response function of such a medium can be found easily in the frequency domain. If we consider a specific frequency component of the electric field at frequency ω, it acquires a phase shift of $\phi = \beta(\omega)L$ as it propagates through such a linear medium of length L, i.e.,

$$\tilde{E}_{\mathrm{out}}(\omega) = \exp[i\beta(\omega)L]\tilde{E}_{\mathrm{in}}(\omega), \tag{4}$$

where $\beta(\omega) = n(\omega)\omega/c$ is the propagation constant of light inside a medium with the refractive index $n(\omega)$ and c is the speed of light in vacuum. It follows from Eq. (2) that $H(\omega) = \exp[i\beta(\omega)L]$. The impulse response function can now be obtained by taking the inverse Fourier transform:

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[i\beta(\omega)L - i\omega t]\mathrm{d}\omega. \tag{5}$$

As a simple check of Eq. (5), we first consider a nondispersive linear medium. In this case, $\beta(\omega) = n_0\omega/c$, where n_0 is the constant refractive index of the medium. By substituting this form of $\beta(\omega)$ back into Eq. (5), the impulse response function for a linear nondispersive medium is found to be

$$h(t) = \delta(t - T_r), \tag{6}$$

where $T_r = n_0L/c$ is the constant transit time of the medium. Equation (6) has a simple intuitive physical interpretation. It shows that a linear nondispersive medium delays each slice of the electric field by a constant transit time of T_r, if we think of the electric field as composed of a sequence of temporal slices.

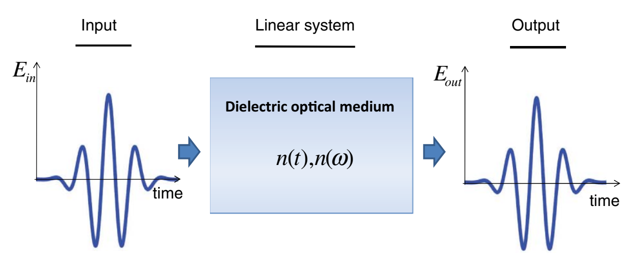

Fig. 1. (Color online) Schematic illustration of the linear system approach to optical pulse propagation. The electric field is shown for both the input and output pulses.

In our previous work [10], we extended this physical interpretation to a dynamic nondispersive medium whose refractive index was allowed to change with time. This was done by allowing the transit time T_r to be different for different temporal slices. Such a simple theory explained quite well the recently discovered phenomenon of adiabatic wavelength conversion that happens in dynamic linear media. We could even extend this approach to a nondispersive nonlinear medium by treating the medium as "linear". By allowing the transit time T_r to depend on local intensity of the pulse, Eq. (6) is able to predict the temporal and spectral features associated with self-phase modulation (SPM) and self-steepening inside a nonlinear Kerr medium [11]. However, dispersive effects were ignored in this previous work.

As seen from Eq. (5), pulse propagation inside a dispersive medium requires knowledge of the frequency dependence of the refractive index since $\beta(\omega) = n(\omega)\omega/c$. A classical harmonic-oscillator model is sometimes used to find this frequency dependence. In an alternative approach, well known in the context of optical fibers [1], $\beta(\omega)$ is expanded in a Taylor series around the carrier frequency ω_0 of the pulse. If we only retain terms up to second order in $\omega - \omega_0$, we can approximate $\beta(\omega)$ as [3]

$$\beta(\omega) \approx \beta_0 + \beta_1(\omega - \omega_0) + \frac{1}{2}\beta_2(\omega - \omega_0)^2, \tag{7}$$

where $\beta_0 = k(\omega_0)$ and β_1 and β_2 are the first- and second-order dispersion coefficients, respectively. Substituting Eq. (7) into Eq. (5) and performing the integration, the impulse response function for such a linear dispersive medium is found to be

$$h(t) = \sqrt{\frac{i}{2\pi\beta_2 L}}\exp\left[\frac{(t - T_r)^2}{2i\beta_2 L} - i\omega_0(t - T_r)\right], \tag{8}$$

where $T_r = \beta_1 L = L/v_g$ is a constant transit time and v_g is the group velocity associated with the pulse.

Strictly speaking, the response function $h(t)$ in Eq. (3) should vanish for $t < 0$ in order to ensure causality [12]. This condition does not hold for $h(t)$ given in Eq. (8), indicating that one must be careful in using it. In spite of this limitation, the Taylor expansion in Eq. (7) is widely used in dealing with nonlinear pulse propagation inside optical fibers, and it has proved quite successful in predicting a wide variety of optical phenomena such as soliton fission and supercontinuum generation [3]. We can justify the use of Eq. (8) heuristically by inserting a factor of $(1 - i\eta)$ in front of the quadrating term containing β_2, where η is a positive number. For any small finite value of η, $|h(t)|$ peaks at $t = T_r$ and becomes negligible for $t < 0$. The use of such convergence factors is often helpful for rapidly oscillating functions.

3. EXTENSION TO A NONLINEAR DISPERSIVE MEDIUM

One important feature of Eq. (8) is that the time variable t is shifted by T_r in both terms on the right side, a feature identical to that appearing in Eq. (6). Although Eq. (8) cannot be simply interpreted as a delay of the electric field slices, the same temporal shift implies that this equation can be extended to include the nonlinear effects by adding the nonlinear slice delay introduced by the medium nonlinearity. As a specific example of the nonlinear medium, we focus on the Kerr nonlinearity and replace $n(\omega)$ with $n(\omega) + n_2 I$, where $I(t)$ is the local intensity of a temporal slice. Similar to the nondispersive case, we include the effect of nonlinearity by replacing T_r with $T_r + T_{\rm nl}$, where $T_{\rm nl} = (n_2 I)L/c$ is the additional delay resulting from the nonlinearity of a Kerr medium [11]. With this change, the impulse response function of a nonlinear dispersive medium can be written as

$$h(t,t') = \sqrt{\frac{i}{2\pi\beta_2 L}} \times \exp\left[\frac{(t-t'-T_r-T_{\rm nl})^2}{2i\beta_2 L} - i\omega_0(t-t'-T_r-T_{\rm nl})\right]. \quad (9)$$

It is important to note that the nonlinear time delay $T_{\rm nl}$ is a function of t' because it depends on the local intensity $I(t')$ of the pulse. Because of this feature, the impulse response function can no longer be written as a single-variable function, rather it is a function both of t and t'.

We can make the impulse response function a single-variable function by using the concept of time transformation introduced in Ref. [11]. For this purpose, we introduce a new time variable t_1 as a nonlinear function of the old time variable t':

$$t_1 = F(t') = t' + T_r + T_{\rm nl}(t'). \quad (10)$$

Clearly, $h(t,t')$ in Eq. (9) can now be written as $h(t-t_1)$. Using the same transformation in Eq. (3), we can write it in the form

$$E_{\rm out}(t) = \int_{-\infty}^{\infty} h(t-t_1)E'(t_1)J(t_1)\mathrm{d}t_1, \quad (11)$$

where $E'(t_1) = E_{\rm in}(t')$ and the Jacobian of the transformation is given by

$$J(t_1) = dt'/dt_1 = (1 + dT_{\rm nl}/dt')^{-1} \quad (12)$$

with $T_{\rm nl}(t') = (n_2 L/c)I(t')$. A major benefit of the time transformation in Eq. (10) is that Eq. (11) is now in the form of a convolution, and the integral appearing there can be calculated efficiently in the Fourier domain using the convolution theorem. A second benefit is that one can include higher-order dispersion terms in Eq. (7) since only the Fourier transform of $h(t)$ is needed to carry our the integration. In fact, dispersion to all orders can be included by using the analytical form this Fourier transform, $H(\omega) = \exp[in \times (\omega)\omega L/c]$.

Before implementing Eq. (11) numerically, we need to discuss one more issue. An implicit assumption made in writing Eq. (9) is that the shape of the pulse does not change considerably during its propagation over the length L. This requirement comes directly from the definition of a linear system that the system should act as a "black box" and is not altered by the "signal" itself. In the nonlinear case, the system is altered by the signal because refractive index of the medium changes locally in proportion to light intensity inside the medium. In practice, pulse shape is not affected much over distances much shorter compared to the dispersion length, defined as $L_D = T_0^2/|\beta_2|$, where T_0 is the pulse width [3]. To cover lengths comparable to or longer than L_D, we adopt an approach similar to that employed by the split-step Fourier method used for solving the NLS equation [3]. Specifically, the nonlinear dispersive medium is divided into multiple sections of length $l \ll L_D$, and Eq. (11) is used to propagate the pulse through each section. At the end of each section, the pulse intensity obtained in that section is used to calculate the functions $E'(t_1)$ and $J(t_1)$ in the next section.

4. PULSES CONTAINING MANY OPTICAL CYCLES

In this section we compare the predictions of our time-transformation approach with those of the standard NLS equation for relatively long pulses containing many optical cycles. To simplify the following discussion, we make use of the expansion in Eq. (7) and include dispersion up to second order through β_2. In general, both the nonlinear parameter n_2 and the dispersion parameter β_2 can take either positive or negative values. Although our technique works for all four possible sign combination of these two parameters, we focus on a medium exhibiting self-focusing nonlinearity $n_2 > 0$ with anomalous dispersion $(\beta_2) < 0$ because optical solitons can form only in this case [3].

The NLS equation deals with the pulse envelope $A(z,t)$ related to the electric field associated with the pulse as

$$E(z,t) = \mathrm{Re}[A(z, t - z/v_g)\exp(i\beta_0 z - \omega_0 t)]. \quad (13)$$

The evolution of $A(z,t)$ inside the dispersive nonlinear medium is then governed by the standard NLS equation written in a normalized form as

$$i\frac{\partial A}{\partial \xi} + \frac{1}{2}\frac{\partial^2 A}{\partial \tau^2} + \beta_0 n_2 L_D |A|^2 A = 0, \quad (14)$$

where $\xi = z/L_D$ is the distance normalized to the dispersion length and $\tau = (t - z/v_g)/T_0$ is the time normalized to input pulse width T_0. If we write the input field in the form $A(0,\tau) = \sqrt{I_0}f_p(\tau)$, where $f_p(\tau)$ governs the pulse shape, we can introduce $N^2 = \beta_0 n_2 I_0 L_D$, where N governs the soliton order [3]. In the following numerical simulations we choose $N = 1$, a condition required for the formation of fundamental solitons that preserve their shape for a specific input pulse shape.

We first consider a Gaussian-shape pulse and use $f_p(\tau) = \exp(-\tau^2/2)$, with $T_0 = 10$ ps at a carrier frequency of 200 THz. Such a pulse contains more than 8000 optical cycles in the spectral region near 1.5 μm relevant for telecommunication systems. Figure 2 compares the predictions of our time-transformation approach with that of the standard NLS equation (14) by plotting the pulse shape and spectrum after a distance of $z = 4L_D$. The input shape and spectrum are shown by the dashed green curves. Both the pulse shape and spectrum change considerably inside the nonlinear medium even when the soliton condition $N = 1$ is satisfied. These changes occur because the the input pulse evolves to become a fundamental soliton with a "sech" pulse shape [3]. The important feature for us is that both the shape and the spectrum obtained with our approach (dotted black curves) coincide perfectly with those obtained with the NLS equation (solid yellow curves), verifying that our method provides a correct description of pulse propagation inside a nonlinear dispersive

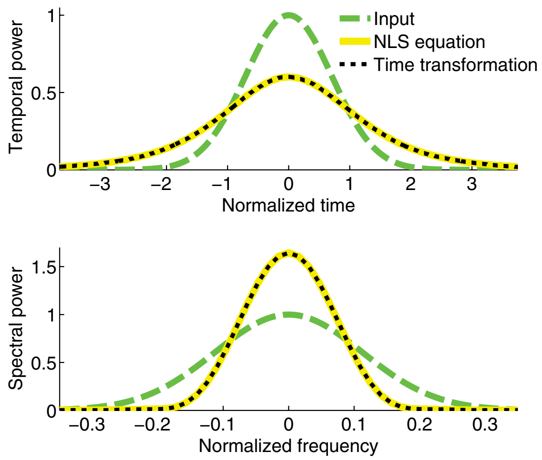

Fig. 2. (Color online) Shape (top) and spectrum (bottom) of a wide optical Gaussian pulse ($T_0 = 10$ ps) at the input end (dashed green lines) and the output end at $z = 4L_D$ (dotted black lines). The predictions of the NLS equation are shown by solid yellow lines.

medium. Although not shown in Fig. 2, our approach propagates the electric field, rather than the pulse envelope, and correctly includes all linear and nonlinear phase shifts.

The most well-known feature of the interplay of nonlinearity and dispersion is the fundamental optical soliton forming when the input pulse has "sech" shape with a peak intensity such that to $N = 1$ [3]. We verify the existence of such a soliton by using $f_p(\tau) = \text{sech}(\tau)$ at $z = 0$. As shown in Fig. 3, the intensity profile of the output pulse (dashed black curve) calculated with our approach overlaps perfectly with that of the input pulse (solid yellow curve) even after propagating a distance of $10L_D$. We verified that the fundamental soliton was able to maintain its shape even for much longer distances.

5. PULSES CONTAINING A FEW OPTICAL CYCLES

In this section we focus the case of pulses that contain just a few optical cycles. For such short pulses, the self-steepening effect, resulting from the intensity-dependent group velocity, becomes important. Also, higher-order dispersion needs to be considered because of a relatively broad spectrum of such pulses. The standard NLS equation (14) cannot provide an accurate description for short optical pulses. However, it can be generalized by adding additional terms that represent

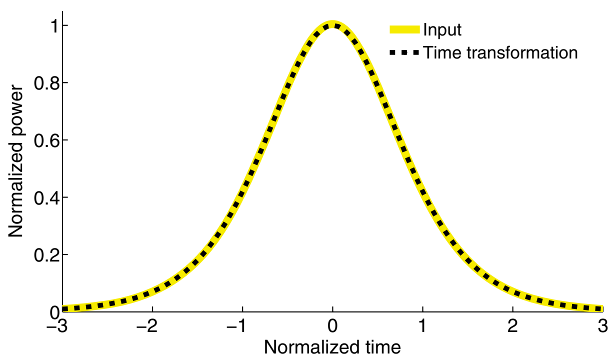

Fig. 3. (Color online) The input (solid yellow) and output (dotted black) intensity profiles at $z = 0$ and $z = 10L_D$ when parameters of the input pulse correspond to a 10 ps fundamental soliton.

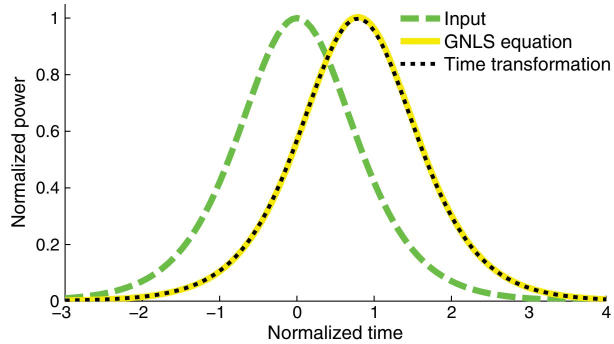

Fig. 4. (Color online) Intensity profile of a sech input pulse at distances of $z = 0$ (dashed green) and $z = 10L_D$ (dotted black) for parameters identical to those of Fig. 3 except for $T_0 = 10$ fs, $s = 1/(4\pi)$, and $\delta_3 = 0.02$. Predictions of the GNLS equation are shown by the solid yellow curve.

the higher-order nonlinear and dispersion effects. Such a generalized NLS (GNLS) equation has the form [3]

$$i\frac{\partial A}{\partial \xi} + \frac{1}{2}\frac{\partial^2 A}{\partial \tau^2} + i\delta_3\frac{\partial^3 A}{\partial \tau^3} + \beta_0 n_2 L_D\left[|A|^2 A + is\frac{\partial}{\partial \tau}(|A|^2 A)\right] = 0, \quad (15)$$

where $\delta_3 = \beta_3/(6T_0|\beta_2|)$ takes into account the third-order dispersion effects governed by β_3 and $s = 1/(\omega_0 T_0)$ is the parameter responsible for self-steepening. We ignore intrapulse Raman scattering in this work.

A question one may ask is whether a fundamental soliton, such as the one shown in Fig. 3, will maintain its shape even in the presence of self-steepening and third-order dispersion. Here we consider a 10 fs fundamental soliton, corresponding to a self-steepening factor of $s = 1/(4\pi)$. We also include third-order dispersion using $\delta_3 = 0.02$. Figure 4 compares the intensity profile at a distance of $z = 10L_D$ with the input one. As seen there, even though the pulse peak moves toward the trailing part, the shape of the pulse remains virtually unchanged with negligible temporal broadening. The shift of the pulse peak results from the influence of intensity-dependent group velocity. The reason that the pulse moves as a soliton is that the third-order dispersive effects in Fig. 4 are not strong enough to break the balance between the SPM and group-velocity dispersion. Notice that the predictions of the GNLS equation agree well with our time-transformation approach.

It is well known that third-order dispersion can lead to fission of higher-order solitons and emission of dispersive waves at a certain well-defined frequency [3]. To verify that our approach predicts these effects as well, we show in Fig. 5 the evolution of a third-order soliton ($N = 3$) by plotting the temporal and spectral profiles of the 10 fs input pulse over a distance of one dispersion length. As seen there, a new spectral peak appears starting at about $z \approx 0.3L_D$ and the pulse splits into three parts soon after that because of the fission of the higher-order soliton induced by third-order dispersion. The frequency shift of $(\nu - \nu_0)T_0 \approx 5$ of the blue shifted peak in Fig. 5 agrees with the phase-matching condition associated with the generation of a dispersive wave. We repeated these simulations using the GNLS equation (15) and found that the results are identical to those shown in Fig. 5.

As a final test of the usefulness of time-transformation approach, we consider a single-cycle pulse. One important

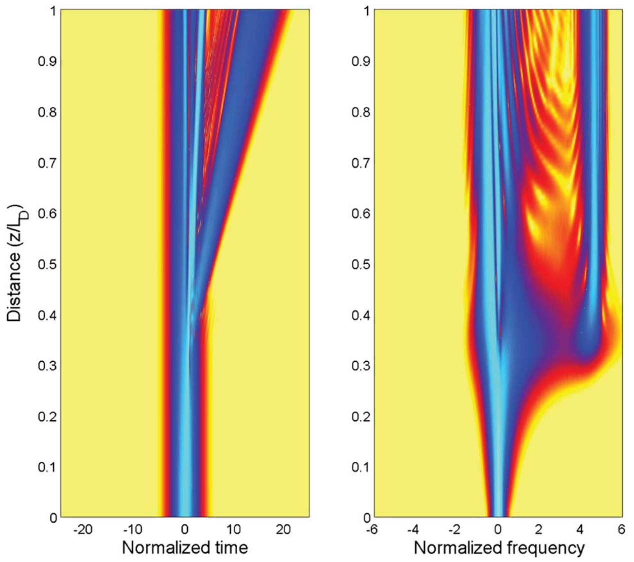

Fig. 5. (Color online) Evolution of a third-order soliton ($N = 3$) over one dispersion length for the same parameters values used in Fig. 4. The temporal and spectral intensities are plotted using a 50 dB color scale.

feature of our approach is that it works directly with the electric field associated with an optical pulse without employing the SVEA. In principle, it is capable of describing the propagation of optical pulses of arbitrary durations. Here, we choose a sech shape pulse with $T_0 = 1$ fs and $\omega_0/(2\pi) = 200$ THz. Such a pulse contains only one optical cycle. The self-steepening parameter now has a relatively large value of about $s = 0.8$, indicating that the electric field of the pulse will be severely distorted during its propagation. For such short pulses, envelope is not well defined and we study how the electric field changes with propagation. We compare our results with those obtained by solving Maxwell's equations directly with the FDTD method. In both cases, the input electric field is taken to be $E_{\rm in}(\tau) = E_0 \,\mathrm{sech}(\tau)\cos(\tau/s)$, with E_0 chosen to ensure $N = 1$. Figure 6 shows changes in the electric field occurring after one dispersion length, where dispersion parameters are identical to those used in Fig. 4. As expected, the electric field is distorted considerably when compared to that at the input. The FDTD result shown by the solid yellow curve indicates that our new approach works

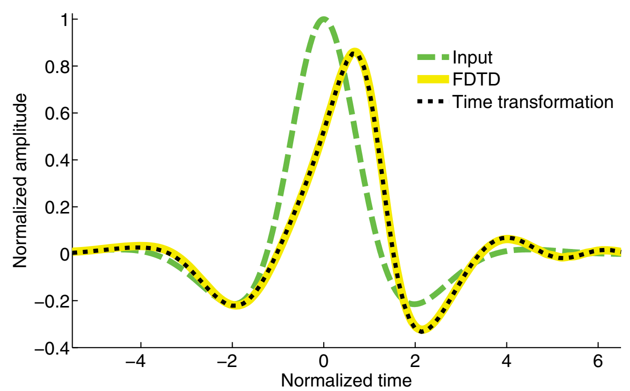

Fig. 6. (Color online) Electric field of a single-cycle pulse (dotted black curve) at $z = L_D$ for $\omega_0/(2\pi) = 200$ THz, $T_0 = 1$ fs, and $N = 1$. Input electric field profile is depicted by the dashed green line. Prediction of the FDTD method is shown by the solid yellow line.

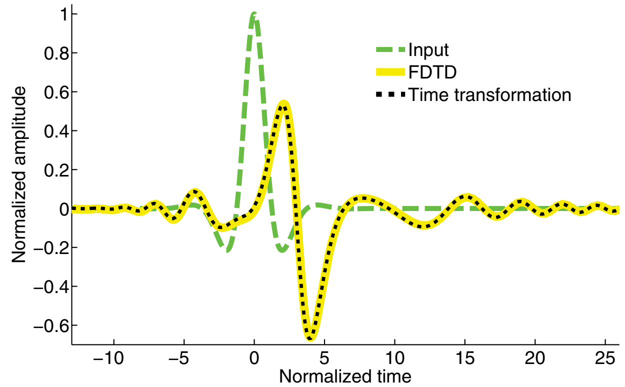

Fig. 7. (Color online) Same as in Fig. 6 but at a distance $z = 5L_D$.

well in the single-cycle regime and provides results numerically much faster than the FDTD technique.

As a further check, Figure 7 shows the electric field after the single-cycle pulse has propagated a distance of $5L_D$. As seen there, compared to the input pulse, the electric field has spread over a much larger temporal range, and its central peak has moved considerably toward the trailing part. These changes indicate that the traditional soliton condition fails in the single-cycle regime because of a strong influence of self-steepening. The FDTD method can also produce the same result, but it requires much more computational resources (by as much as a factor of 100 in the case of Fig. 7).

One may ask whether the GNLS equation (15) can be used for a single-cycle pulse. As discussed in Ref. [4], this equation can be used for pulses containing a few optical cycles. Indeed, our numerical results confirm that it can be applied to pulses containing a single cycle. Computational effort required by our technique is comparable to that required for solving Eq. (15).

6. CONCLUDING REMARKS

In conclusion, we have applied our time-transformation approach, first developed in the context of a dynamic, linear, nondispersive medium [10], for studying propagation of short optical pulses inside a nonlinear dispersive medium. We extend the linear system theory so that it can also be used for a nonlinear medium using the time-transformation concept. A nonlinear temporal mapping permits us to include dispersion to all orders by using an arbitrary form of the frequency dependence of the refractive index. This new intuitive approach shows very good agreement with other techniques based on the GNLS equation and the FDTD method in the appropriate limit. A major benefit of our approach over the FDTD method is that it is much faster computationally and can be used for propagating optical fields over longer distances. As an example, the propagation distance in Fig. 7 is 5 mm for $T_0 = 1$ fs if we use $\beta_2 = -1$ ps^2/km as a typical value. Since our time-transformation approach does not make use of the SVEA, it can also be used for pulses of any duration.

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