Nonlinear pulse propagation: A time–transformation approach

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We present a time-transformation approach for studying the propagation of optical pulses inside a nonlinear medium. Unlike the conventional way of solving for the slowly varying amplitude of an optical pulse, our new approach maps directly the input electric field to the output one, without making the slowly varying envelope approximation. Conceptually, the time-transformation approach shows that the effect of propagation through a nonlinear medium is to change the relative spacing and duration of various temporal slices of the pulse. These temporal changes manifest as self-phase modulation in the spectral domain and self-steepening in the temporal domain. Our approach agrees with the generalized nonlinear Schrödinger equation for 100 fs pulses and the finite-difference time-domain solution of Maxwell's equations for two-cycle pulses, while producing results 20 and 50 times faster, respectively.

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An optical pulse modulates its own phase when passing through a nonlinear medium because of an intensity-dependent refractive index. This kind of phase modulation, known as self-phase modulation (SPM), was discovered in 1967 and manifested as spectral broadening of optical pulses after passing through a nonlinear Kerr medium [1]. The observed asymmetry in both the broadened spectrum and the temporal shape for ultrashort pulses was caused by self-steepening occurring because of an intensity-dependent group velocity [2]. The conventional method for calculating these nonlinear phenomena is the nonlinear Schrödinger (NLS) equation, obtained from Maxwell's equations under the slowly varying envelope approximation and is accurate as long as the input pulse is considerably wider than the optical period [3]. A generalized NLS equation is developed and can be applied in the single-cycle regime [4].

In this Letter, we present a novel time-transformation approach to study the propagation of optical pulses through a nonlinear medium. Specifically, we extend our approach initially applied to the dynamic linear medium in [5] to the case of a nonlinear medium and apply it to study the impact of SPM and self-steepening on optical pulses. Our approach maps directly the input electric field to the output one, without making the slowly varying envelope approximation. It clearly reveals that self-steepening is an integral part of the SPM process, both of which result from the intensity dependence of the refractive index. We apply the time-transformation approach to the specific case of a nondispersive Kerr medium for the first time to show that it agrees with the conventional NLS approach in the appropriate limit.

Pulse propagation through a dynamic optical medium can be described as a generalized convolution that relates the output electric field $E_{\rm out}(t)$ to the input $E_{\rm in}(t)$ [5]:

$$E_{\rm out}(t) = \int_{-\infty}^{\infty} \delta[t - t' - T_r(t')] E_{\rm in}(t') \mathrm{d}t', \qquad (1)$$

where $\delta[t - t' - T_r(t')]$ is the impulse response function for a nondispersive medium. We interpret this relation as a temporal mapping of each input slice located at t' to a corresponding output slice located at t via

$$t = t' + T_r(t'), \qquad (2)$$

where $T_r(t')$ is the transit time associated with the slice of the input pulse at time t'. We refer to the temporal mapping in Eq. (2) as a time-transformation; this time-transformation is performed directly on the electric field via Eq. (1). We used such a temporal-mapping approach in [5] to describe adiabatic wavelength conversion, a process that occurs in a dynamic linear medium (whose refractive index changes with time) [6]. Here, we show that the time-transformation approach can be applied to a nonlinear medium as well. More specifically, the nonlinear medium is treated as being linear, but its refractive index changes with time as dictated by the pulse intensity profile. Such an approach is similar to that used in [7] in the context of nonlinear optical waveguides.

Unlike the NLS equation that discards the rapid temporal oscillation of the electric field in favor of the slowly varying pulse envelope, our approach maps directly the input electric field into the output one. As a specific nonlinear optics example, we focus on a nondispersive Kerr medium whose refractive index $n(t) = n_0 + n_2 I(t)$ is separated into a linear part n_0 and a nonlinear part that is the product of the Kerr nonlinearity n_2 and the intensity $I(t)$. A direct consequence of such an intensity-dependent refractive index is that different electric field slices of the pulses are transported through the medium at different speeds, $\nu(t) = c/n(t)$, depending on the local value of refractive index $n(t)$. In other words, the transit time of a slice depends on the local intensity $I(t) = I_0 f(t)$ of the pulse as

$$T_r(t') = T_{r0} + n_2 I_0 f(t') L/c, \qquad (3)$$

where I_0 is the peak intensity of a pulse with shape $f(t)$, L is the medium length, c is the speed of light in vacuum, and the constant part, $T_{r0} = n_0 L/c$.

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Figure 1(a) shows the time-transformation performed by a Kerr medium for a Gaussian pulse having $[f(t) = \exp(-t^2/T_0^2)]$ using $T_{r0}/T_0 = 2$ and $\tau_n/T_0 = 0.8$, where $\tau_n = n_2 I_0 L/c$ is the maximum nonlinear delay experienced by the pulse. The spacing of the time slices is determined by the scaling factor s, where $s = dt'/dt$. In the case of a linear medium ($n_2 = 0$), the 45° slope indicates that the input and output time arrays are spaced the same, corresponding to a linear time transformation. However, in the case of a nonlinear medium, the slope varies along the pulse and the mapping becomes nonlinear; in this case, the transit time varies from one slice to the next. For $n_2 > 0$, slices near the central part of the input pulse move towards the trailing edge because of their reduced speed, resulting a distortion of the pulse shape. For $n_2 < 0$, distortion occurs as slices near the central part of the input pulse move towards the leading edge because of their increased speed. This kind of pulse distortion is known as self-steepening, resulting from intensity dependence of the group velocity [3].

The nonlinear changes in the slice spacing also leads to corresponding changes in the instantaneous frequency of the pulse, which is proportional to the scaling factor s. For $n_2 > 0$, temporal stretching occurs in the front part of the pulse, leading to a corresponding decrease in s, as depicted by the solid line in Fig. 1(b), which is the familiar frequency red-shift. In contrast, temporal compression occurs in the back portion of the pulse, resulting in a higher value of s and instantaneous frequency (blue-shift). The opposite behavior occurs for a negative Kerr medium ($n_2 < 0$) as shown in Fig. 1; for the case of linear medium, $s = 1$ and no new frequencies are generated. This variation in frequency is the origin of SPM in our physical approach. Creation of the new frequency components (frequency chirping) is a simple consequence of the nonlinear temporal stretching or compression of the electric field slices during the temporal mapping.

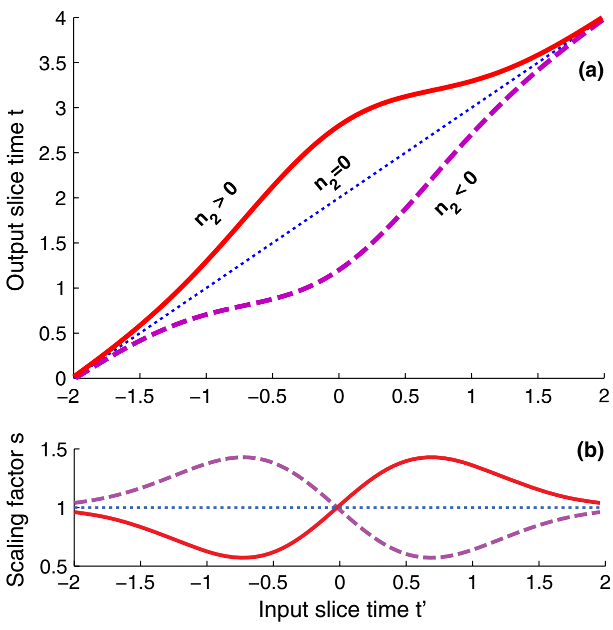

Fig. 1. (Color online) (a) Time-transformation performed by a medium for Gaussian pulse propagation and (b) the corresponding scaling factor s. Cases: $n_2 > 0$ (solid); $n_2 < 0$ (dashed); $n_2 = 0$ (dotted).

Substituting the Kerr medium form of the transit time $Tr(t')$ in Eq. (3) into Eq. (1), the output electric filed is found to be

$$E_{\text{out}}(t) = \int_{-\infty}^{\infty} \delta[t - t' - T_{r0} - \tau_n f(t')] E_{\text{in}}(t') \mathrm{d}t'. \quad (4)$$

The δ function in Eq. (4) performs two actions. First, it performs a change of variables dictated by Eq. (2). Second, it relates the value of E_{out} to E_{in} as

$$E_{\text{out}}(t) = [dt'/dt] E_{\text{in}}(t') = s(t') E_{\text{in}}(t'), \quad (5)$$

where the scaling factors s is the appropriate Jacobian. In numerical calculations, the output time array t is first generated by the input t' following Eqs. (2) and (3). The output electric field is then assigned according to Eq. (5). Thus, given an input electric field, the output field can be obtained without any numerical integration.

This simple mapping matches the pulse shape and spectrum predictions based on the standard approach using the NLS equation. In Figs. 2(a) and 2(c), we consider propagation of a relatively wide 10 ps Gaussian pulse ($f_0 = 200$ THz) through a Kerr medium with a peak intensity such that $\phi_m = 2\pi f_0 \tau_n = 20$, where ϕ_m is the maximum SPM-induced phase shift [3]. As shown in part 3(a), the pulse shape does not change much (the input pulse shape falls under the plotted output shape). However, the output spectrum in part 3(c) shows considerable SPM-induced broadening. The dashed curves show the predictions of the NLS equation, which takes the following form in the absence of dispersive effects:

$$\frac{\partial A}{\partial z} = i\gamma |A|^2 A, \quad (6)$$

where $A(z, t)$ is the envelope of the pulse and $\gamma = n_2 \omega_0 / c$ is a nonlinear parameter [3]. Predictions from the NLS equation and our approach are indistinguishable.

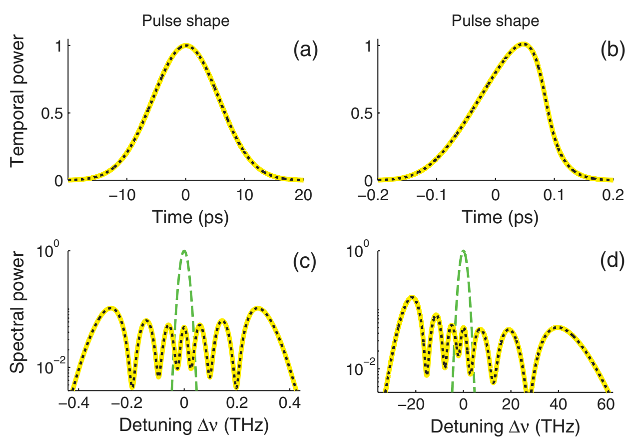

Fig. 2. (Color online) Output shape and spectrum of 10 ps (left column) and 100 fs (right column) Gaussian pulse obtained with our approach (solid lines) and using the NLS equation (dotted lines). Dashed curve shows input spectra in each case.

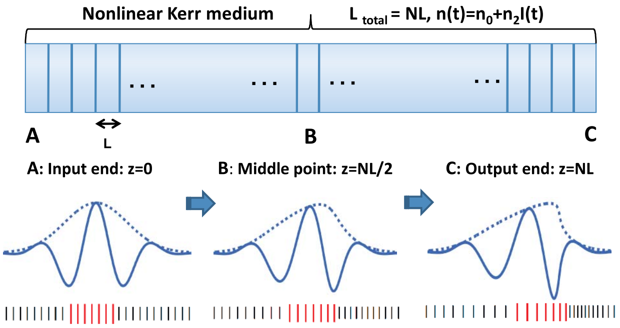

Fig. 3. (Color online) Schematic propagation of the electric field of a 2-cycle pulse through a Kerr medium ($n_2 > 0$). Dotted curves show pulse envelope. Red time slices at the bottom move toward the back of the pulse.

The direct application of Eq. (4), however, assumes that the pulse shape does not change considerably during its propagation inside the medium, an assumption that does not hold for short pulses. As shown in Fig. 3, to account for shape changes, we divide the whole medium into small sections and apply Eq. (4) in a stepwise fashion from one section to another, which is similar to the *split-step Fourier method* used commonly for nonlinear optical problems [3]. More specifically, the output pulse shape obtained at the end of one section is used as the intensity profile seen by the next section. Note that, for this stepwise case, the length L in Eq. (4) is the length of a small section and the whole medium length $L_{\text{total}} = NL$, for a medium divided into N sections.

Solid (yellow) curves in Figs. 2(b) and 2(d) show the predicted shape and spectrum of 100 fs Gaussian pulses using Eq. (4); all other parameters are identical to those in 2(a) and 2(c). As seen in part 2(b), the pulse shape is distorted considerably through self-steepening. This distortion is due to the nonuniform stretching or compression of various temporal slices. The asymmetric broadening of the spectrum, depicted in part 2(d), represents the impact of self-steepening on SPM. The number of medium slices was increased until the output power profile converged; 40 slices were required for the chosen nonlinearity strength here ($\tau_n/T_0 = 0.2$).

The standard approach for calculating the propagation of ultrashort pulses uses a generalized version of Eq. (6) that includes the self-steepening effect by adding an additional term as follows [3]:

$$\frac{\partial A}{\partial z} = i\gamma|A|^2 A - \frac{\gamma}{\omega_0}\frac{\partial |A|^2 A}{\partial t}. \tag{7}$$

The results of Eq. (7) are shown by dotted lines in Figs. 3(b) and 3(d) and match our time-transformation approach, but took 20 times longer to produce.

Finally, we consider the nonlinear propagation of few-cycle pulses. The solid curve in Fig. 4 shows the output electric field of a 2-cycle Gaussian pulse (5 fs, $f_0 = 200$ THz) using our approach ($N = 45$). Note in this case, ϕ_m is chosen to be a relatively small value of 1 so that shock does not form. As shown in Fig. 4, the nonlinear stretching and compression of the electric field in the front and back part of the pulse are clearly seen. Numerical calculation is also performed using the finite-difference time-domain (FDTD) solution of Maxwell's equations [8], as depicted by the dotted curve. The excellent agreement with FDTD method shows that our approach is a valid approach to study few-cycle pulse propagation. What is more important is that our approach is over 50 times faster.

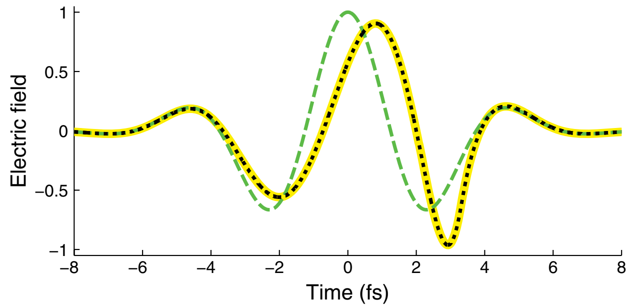

Fig. 4. (Color online) Nonlinear propagation of a 2-cycle Gaussian pulse (solid: time-transformation; dotted: FDTD). Dashed curve shows the input electric field.

In conclusion, we have presented a novel time-transformation approach for studying pulse propagation in nonlinear media. Its application to a Kerr medium shows that the effect of propagation is to change the relative temporal locations of the individual electric field slices. We hope to extend this method to include dispersion and finite nonlinear response time so that it can be applied to study propagation of ultrashort pulses in a variety of nonlinear dispersive media.

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