Nonlinear Pulse Evolution in Silicon Waveguides: An Approximate Analytic Approach

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***Abstract*—Owing to recent progress in silicon-on-insulator (SOI) technology for signal processing of optical pulses, a detailed intuitive understanding of the different processes governing pulse propagation through SOI waveguides is desired. Even though it is possible to carry out numerical simulations to characterize device performance by varying material and pulse parameters, such an approach does not provide an intuitive understanding. For this reason, we develop an analytic approach in this paper and present approximate solutions that are valid under realistic conditions and characterize with reasonable accuracy the dynamical evolution of a short optical pulse through SOI waveguides. Our analytical expressions take into account linear losses, Kerr nonlinearity, two-photon absorption, and free-carrier effects (both absorptive and dispersive) and thus are likely to be useful for a variety of applications in the area of silicon photonics. Even though free-carrier absorption is included, we limit our analysis to the case where its influence on the temporal pulse shape is minimal. To provide a comprehensive understanding of our results and to validate their accuracy, we consider general properties of our analytical solutions, analyze their applicability in different parametric ranges relevant for applications, and compare them with published results. We envision utilizing these results in optimizing the design of SOI-based devices aimed at integrated optics applications.**

***Index Terms*—Free-carrier absorption, free-carrier dispersion, optical Kerr effect, optical pulse propagation, silicon waveguides, two-photon absorption.**

I. Introduction

OWING to the presence of several simultaneous nonlinear phenomena, optical pulse propagation in silicon-on-insulator (SOI) waveguides is a complicated physical process with many interesting features and applications [1]–[5]. Because the crystalline silicon possesses inversion symmetry, it does not exhibit second-order nonlinear effects. Even though there are ways of breaking this inversion symmetry to initiate the second-order electro-optic effects in silicon [6], the dominant nonlinear response arises from the third-order susceptibility. Important contributions to this susceptibility tensor result from the third-order electronic (Kerr) and phonon (Raman) nonlinearities (nearly two and four orders of magnitude stronger than those of silica glass, respectively) and the two-photon absorption (TPA) process. Moreover, TPA creates free carriers whose presence leads to additional losses through free-carrier absorption (FCA) and refractive-index changes through free-carrier dispersion (FCD). In this paper, we focus on the nonresonant nonlinearities and include the Kerr, TPA, FCA, and FCD effects but exclude the Raman effect because of its resonant nature in silicon. Interestingly, these nonresonant effects are further enhanced inside SOI waveguides because of a tight mode confinement leading to a relatively small effective mode area ($<1\ \mu m^2$). Further, the dispersion inside SOI waveguides can be easily tailored by adjusting the waveguide geometry [7]–[10]. All these amalgamate to a material with rich optical features and great potential for structural and performance engineering, making silicon a viable alternative and a potential contender for integrated optics applications [11]–[13].

Renewed interest and high level of activity in silicon photonics technology can be gauged by looking at plethora of recently reported applications. For example, SOI waveguides have been successfully employed for supercontinuum generation [14], [15], soliton formation [16], [17], pulse compression and mode-locking [18], wavelength conversion [19], [20], high-speed optical switching [21], Raman amplification [2], and Raman lasing [22]–[24]. The basis for such useful applications is mainly provided by the optical Kerr effect and stimulated Raman scattering, whereas TPA and FCA usually limit device operation due to strong absorption and resulting attenuation of the optical signal [25]–[28]. In order to control these detrimental effects and to better understand the potential of silicon as a nonlinear material, various nonlinear effects such as self-phase modulation, cross-phase modulation, and four-wave mixing have been intensively studied, both experimentally and theoretically [29]–[35].

As a consequence of these studies, at present, we have a set of general equations that provide a comprehensive theoretical description of the nonlinear response of SOI waveguides [36], [37]. Most importantly, they can be used to simulate the propagation of intense ultrashort (up to several cycles) electromagnetic pulses. Unfortunately, these equations are analytically intractable due to their complex mathematical structure. However, simple analytical solutions are of considerable value since they provide a comprehensive picture of different interacting processes and related parameters and eventually pave the path

Manuscript received October 09, 2008; revised December 27, 2008. First published April 24, 2009; current version published July 20, 2009. This work was supported by the Australian Research Council through its Discovery Grant scheme. The work of G. P. Agrawal was also supported by The NSF Award ECCS-0801772.

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Digital Object Identifier 10.1109/JLT.2009.2012412

for innovation and design optimization. Moreover, such strategies have a pedagogical value because the same mathematical technique may be used for solving similar sets of equations in other disciplines. For example, we have recently applied a multiple-scale technique to construct analytical solutions of optical pulses propagating through semiconductor optical amplifiers [43] and used the results for understanding the operation of several useful devices [44], [45].

Even though many numerical studies have been done in the past, the only exact analytical solution for pulse propagation through SOI waveguides was reported in [38] after excluding the influence of Raman, FCA, and FCD effects. Although the neglect of the Raman effect may be justified in the absence of a pump beam, this cannot be said for the FCA and FCD effects, whose omission places severe constraints on the maximum value of the pulse energy. In many realistic situations, free-carrier effects are of fundamental importance because they introduce substantial qualitative differences for SOI waveguides compared with optical fibers in which free carriers play no role. Therefore, it is desirable to construct an analytical description of the nonlinear evolution of optical pulses in SOI waveguides that includes the impact of free carriers. The main aim of this paper is to derive useful analytical expressions describing pulse propagation through SOI waveguides under realistic situations. This paper is organized as follows. In Section II, we establish the notation, present the coupled system of differential equations that govern pulse evolution in SOI waveguides, and recast them in a form suitable for constructing analytical solutions. In Section III, we derive an approximate analytical solution, valid in a broad range of parameter values. Inspired the by the solution given in [38], this general solution is then investigated under various simplifying assumptions in Section IV to derive analytical expressions with a limited validity but providing much insight in the process. In Section V, we conduct a comprehensive numerical study to analyze the validity of our analytical solution. We conclude this paper in Section VI after summarizing the results.

II. Nonlinear Model of Pulse Propagation

Pulse propagation through SOI waveguides can be characterized quite accurately within the framework of the widely used slowly varying envelope approximation. If we ignore the dispersive effects, its use leads to the following set of coupled nonlinear partial differential equations for characterizing optical pulses interacting with optically excited carriers within the SOI waveguide [36], [38]–[40]:

$$\frac{\partial A}{\partial z} = -\frac{\alpha}{2}A - k|A|^2A - qNA \tag{1}$$

$$\frac{\partial N}{\partial \tau} = -\frac{N}{\tau_c} + p|A|^4. \tag{2}$$

Here $N(z,\tau)$ is the carrier density and $A(z,\tau)$ is the pulse envelope (in the units of intensity$^{1/2}$) related to the electric field as

$$\boldsymbol{E}(\boldsymbol{r},\tau) = \hat{e}\,\varpi\,F(x,y) \times [A(z,\tau)\exp(i\beta_0 z - i\omega_0\tau) + \text{c.c.}]$$

where ω_0 is the carrier frequency, $\beta_0 = n_0\omega_0/c$ is the propagation constant, n_0 is the effective refractive index of the optical mode with the spatial profile $F(x,y)$ and polarized along the unit vector $\hat{e}, \varpi = (\mu_0/\varepsilon_0)^{1/4}(2n_0)^{-1/2}$ is the dimension factor. Note that $\tau = t - z/v_g$ is the retarded time in the reference frame of the pulse and v_g is the group velocity.

Among the parameters appearing in (1), α is the linear loss coefficient, $k = \beta/2 - i\gamma$ results from the third-order susceptibility, β is the TPA coefficient, and $\gamma = \omega_0 n_2/c$ is the nonlinear Kerr parameter. The parameter $q = q_r + iq_i$ governs the impact of free carriers. Its real part, $q_r = (\sigma_\alpha/2)(\lambda_0/\lambda_r)^2$, accounts for FCA, and its imaginary part, $q_i = (\omega_0/c)\,\sigma_n(\lambda_0/\lambda_r)^2$, accounts for the FCD effects. Here $\sigma_\alpha = 1.45 \times 10^{-21}$ m^2, $\sigma_n = 5.3 \times 10^{-27}$ m^3, and $\lambda_r = 1550$ nm [36]. The two parameters appearing in (2) are real; τ_c is the effective carrier lifetime and $p = \beta/(2\hbar\omega_0)$.

The assumptions used in deriving (1) and (2) can be found in the recent review of Lin *et al.* [36]. The most important among them are the neglect of stimulated Raman scattering and of group-velocity dispersion. The former is valid when the the pulse bandwidth is shorter than the Raman shift (15.1 THz for silicon). The latter holds for waveguides much shorter than the dispersion length, $L_d = T_0^2/|\beta_2|$, where T_0 is the temporal pulse width and β_2 is the second-order dispersion parameter. In what follows we assume that these conditions hold for the SOI waveguide under consideration. Among the above-listed assumptions, (1) assumes that the pulse is polarized along the fundamental TE or TM mode of the waveguide and therefore maintains its initial state of polarization. This allows one to characterize the electronic susceptibility of silicon by only using the component $\chi^e_{1111}(-\omega;\omega,-\omega,\omega)$, which can be expressed through TPA coefficient β and nonlinear Kerr coefficient n_2. For the numerical examples of Section V, we use the following parameter values: $\lambda_0 = 1550$ nm, $n_0 = 3.484, \alpha = 1$ dB/cm, $\beta = 5 \times 10^{-12}$ mW, $n_2 = 6 \times 10^{-18}$ m^2 W, $\tau_c = 1$ ns, $|\beta_2| = 1$ ps^2 m.

It is convenient to rewrite (1) and (2) in the terms of a real amplitude, $a(z,\tau)$, and a real phase, $\phi(z,\tau)$, of the pulse envelope. Assuming that

$$A(z,\tau) = a(z,\tau)\exp[i\phi(z,\tau) - (\alpha/2)z]$$

we obtain a set of following three differential equations:

$$\frac{\partial a}{\partial z} = -\frac{\beta}{2}\,a^3 \mathrm{e}^{-\alpha z} - q_r N a \tag{3}$$

$$\frac{\partial \phi}{\partial z} = -q_i N + \gamma\, a^2 \mathrm{e}^{-\alpha z} \tag{4}$$

$$\frac{\partial N}{\partial \tau} = -\frac{N}{\tau_c} + p\, a^4 \mathrm{e}^{-2\alpha z}. \tag{5}$$

It is useful to gain a qualitative feel of the physical essence of these equations before proceeding further. Equation (3) accounts for the attenuation induced by the both TPA and FCA processes. Equation (4) governs the nonlinear phase changes due to the Kerr and FCD effects as the pulse propagates along the SOI waveguide. Equation (5) describes the creation of electron-hole pairs through TPA and their subsequent reduction via electron-hole recombination or diffusion out of the mode area. Note that the free carriers affect the pulse intensity only through FCA, but the pulse phase is affected by both the FCA (through a^2) and FCD (through q_i). It is interesting to note that even though (3) and (5) are coupled, (4) does not have such dependency and hence can be solved by direct integration once a solution for those coupled equations is found. Unfortunately, an exact solution of (3) and (5) cannot be obtained analytically because their coupling is intrinsic and not separable. In Section III, we concentrate on devising a strategy to construct an approximate solution for the coupled system (3)–(5) with a widest range of applicability.

III. Approximate Solution of the Propagation Equations

We start our derivation by constructing a single integro-differential equation by combining (3) and (5). Noting that (5) can formally be solved as

$$N(z,\tau) = N_0(z) + p\,\mathrm{e}^{-2\alpha z}\int_{-\infty}^{\tau} \mathrm{e}^{-(\tau-\tau')/\tau_c} a^4(z,\tau')\,d\tau' \quad (6)$$

and substituting this result into (3), we obtain

$$\frac{\partial a}{\partial z} = -\frac{\beta}{2}a^3\mathrm{e}^{-\alpha z} - q_r N_0 a - \chi a\,\mathrm{e}^{-2\alpha z}\int_{-\infty}^{\tau} \mathrm{e}^{-(\tau-\tau')/\tau_c} a^4(z,\tau')\,d\tau' \quad (7)$$

where $\chi = q_r p$, and $N_0(z)$ is the constant (background) carrier density that may exist before the pulse enters the waveguide. To simplify the following analysis, we assume that $N_0(z) = 0$. However, we stress that this simplification has no bearing on the final results or integrability of intermediate expressions if one chooses to keep this term.

The first term on the right side of (7) accounts for TPA, while the last term is responsible for FCA. For the parameters of practical interest, these terms have comparable magnitudes, and thus a standard perturbation expansion (e.g., regular perturbation method [42]) cannot be applied. To proceed further, we assume that the free carriers generated by TPA modify the temporal shape of the pulse only slightly. With this assumption, (7) can be approximated as

$$\frac{\partial a}{\partial z} \approx -\frac{\beta}{2}a^3\mathrm{e}^{-\alpha z} - \chi a\,\mathrm{e}^{-2\alpha z}\int_{-\infty}^{\tau} \mathrm{e}^{-(\tau-\tau')/\tau_c} a_0^4(z,\tau')\,d\tau' \quad (8)$$

where $a_0(z,\tau)$ is the solution of (7) with $\chi = 0$ and is given by

$$a_0(z,\tau) = \frac{u(\tau)}{\sqrt{1 + (\beta/\alpha)\,u^2(\tau)[1-\exp(-\alpha z)]}}$$

where $u(\tau)$ is the temporal profile of the input pulse.

To solve (8), we use the following ansatz for describing the evolution of pulse amplitude through the SOI waveguide:

$$a(z,\tau) = \frac{a_0(z,\tau)}{\sqrt{1+g(z,\tau)}}. \quad (9)$$

Clearly, the unknown function $g(z,\tau)$ should satisfy the boundary condition $g(0,\tau) = 0$. Substituting (9) into (8), we obtain the following equation for $g(z,\tau)$:

$$\frac{\partial g}{\partial z} = -\beta a_0^2 g\,\mathrm{e}^{-\alpha z} + 2\chi(1+g)\,\mathrm{e}^{-2\alpha z}\int_{-\infty}^{\tau} \mathrm{e}^{-(\tau-\tau')/\tau_c} a_0^4(z,\tau')\,d\tau'. \quad (10)$$

This is a first order, linear, inhomogeneous differential equation in the variable g with an integrating factor. Thus, we can easily write the solution for this equation in the form

$$g(z,\tau) = a_0^2(z,\tau)\,\mathrm{e}^{v(z,\tau)}\int_0^z \mathrm{e}^{-v(z',\tau)} s(z',\tau)\,dz' \quad (11)$$

where $v(z,\tau)$ and $s(z,\tau)$ are yet to be determined. Substitution of (11) into (10) gives

$$g\frac{\partial v}{\partial z} + a_0^2 s = 2\chi(1+g)\,\mathrm{e}^{-2\alpha z}\int_{-\infty}^{\tau} \mathrm{e}^{-(\tau-\tau')/\tau_c} a_0^4(z,\tau')\,d\tau'.$$

This equation will be identically true if the following relations are satisfied:

$$\frac{\partial v}{\partial z} = a_0^2 s = \zeta(z,\tau) \equiv 2\chi\,\mathrm{e}^{-2\alpha z}\int_{-\infty}^{\tau} \mathrm{e}^{-(\tau-\tau')/\tau_c} a_0^4(z,\tau')\,d\tau'.$$

Thus, the functions $s(z,\tau)$ and $v(z,\tau)$ are found to be

$$s(z,\tau) = \frac{\zeta(z,\tau)}{a_0^2(z,\tau)}$$

$$v(z,\tau) = \int_0^z \zeta(z',\tau)\,dz' = 2\chi w(z,\tau)$$

where

$$w(z,\tau) = \int_0^z \mathrm{e}^{-2\alpha z'}\int_{-\infty}^{\tau} \mathrm{e}^{-(\tau-\tau')/\tau_c} a_0^4(z',\tau')\,d\tau' dz' = \int_{-\infty}^{\tau} \mathrm{e}^{-(\tau-\tau')/\tau_c}\varphi(z,\tau')\,d\tau'$$

$$\varphi(z,\tau) = \frac{a_0^2}{\beta}\Big(1+\frac{\beta}{\alpha}u^2\Big)(1-\mathrm{e}^{-\alpha z}) - \frac{\alpha}{\gamma\beta}\phi_0$$

$$\phi_0(z,\tau) = \frac{\gamma}{\beta}\ln\left[1+\frac{\beta}{\alpha}u^2(1-\mathrm{e}^{-\alpha z})\right].$$

An approximate solution of (4) can be found in the same manner. Substituting (6) into (4) we obtain

$$\frac{\partial \phi}{\partial z} \approx \gamma a\,\mathrm{e}^{-\alpha z} - \xi\,\mathrm{e}^{-2\alpha z}\int_{-\infty}^{\tau} \mathrm{e}^{-(\tau-\tau')/\tau_c} a_0^4(z,\tau')\,d\tau'$$

where $\xi = q_i p$, and a is given by (9). The solution of this equation is found to be

$$\phi(z,\tau) = \gamma\int_0^z \frac{a_0^2(z',\tau)}{1+g(z',\tau)}\,\mathrm{e}^{-\alpha z'}dz' - \xi\,w(z,\tau). \quad (12)$$

Equations (6), (9), (11), and (12) provide an approximate solution of the nonlinear system (3)–(5) along the SOI waveguide. It is easy to verify that, in the limit $\chi \to 0$ and $\xi \to 0$ (i.e., no FCA and FCD), we recover the well-known result

$a = a_0(z,\tau), \phi = \phi_0(z,\tau)$ [38]. In the general case, several qualitative conclusions can be drawn from the structure of our solution. The FCA and FCD effects are manifested as cumulative effects passed over by the integral functions $\zeta(z,\tau)$ and $w(z,\tau)$. They are responsible for the asymmetry of the amplitude and phase of an initially symmetric pulse. Second, as one would expect, $g(z,\tau) \geq 0$, indicating that the inclusion of FCA leads to a reduction in the pulse amplitude, and a corresponding additional energy loss, compared with the TPA-only scenario. Finally, $w(z,\tau) \geq 0$ as well. Thus, as is well known [38], FCD reduces the phase shift associated with the Kerr nonlinearity. As we shall see in Section V, this reduction can be sufficiently large even when the pulse shape changes slightly.

IV. Particular Cases of the Analytic Solution

It is instructive to gain an in-depth understanding of the analytic results by focusing on several distinct simplified scenarios. Even though some of these scenarios may not necessarily reflect a practically or physically significant situation, the overall insight and understanding they provide cannot be underestimated.

A. Approximation Leading to Isolation of FCA and FCD Effects

Equations (11) and (12) can be substantially simplified and analyzed in more detail if we assume that

$$\delta \equiv \max[v(z,\tau)] \ll 1. \tag{13}$$

As discussed in Section V, this inequality follows from the usability condition for our solution for over a wide range of parameters for SOI waveguides of practical interest. Neglecting the function $g(z',\tau)$ in the denominator of (12) [see condition (19)] and using (13) we obtain the following result:

$$\begin{aligned} g(z,\tau) &\approx 2\chi a_0^2(z,\tau)\int_0^z a_0^{-2}(z',\tau)\,\mathrm{e}^{-2\alpha z'} \\ &\quad\times \int_{-\infty}^{\tau} \mathrm{e}^{-(\tau-\tau')/\tau_c} a_0^4(z',\tau')\,d\tau' dz' \\ &= 2\chi a_0^2(z,\tau)\int_{-\infty}^{\tau} \mathrm{e}^{-(\tau-\tau')/\tau_c}\psi(z,\tau,\tau')\,d\tau' \end{aligned} \tag{14}$$

$$\phi(z,\tau) \approx \phi_0(z,\tau) - \xi\, w(z,\tau) \tag{15}$$

where

$$\begin{aligned} \psi(z,\tau,\tau') = \frac{1}{\beta}&\left[\left(1-\frac{u^2(\tau')}{u^2(\tau)}\right)\right. \\ &\left.\times\left(\frac{a_0^2(z,\tau')}{u^2(\tau')}\mathrm{e}^{-\alpha z}-1\right)+\mathrm{e}^{-\alpha z}-1\right] \\ &+\frac{1}{\gamma}\left[\frac{\alpha}{\beta}\left(\frac{2}{u^2(\tau')}-\frac{1}{u^2(\tau)}\right)+1\right]\phi_0(z,\tau'). \end{aligned}$$

Now we are able to consider the impact of free-carriers more elaborately. The manner in which FCA and FCD cause changes in the pulse amplitude a and phase ϕ is clearly indicated by (14) and (15). Whether the input pulse-shape function $u(\tau)$ is even or odd, the functions $g(z,\tau)$ and $w(z,\tau)$ do not possess a definite parity with respect to τ, thus leading to an asymmetrical evolution of both the pulse shape and spectrum along the SOI waveguide.

Two limiting cases of ultrashort and ultrawide pulses provide further insight into the role of free carriers. Consider first the case of pulses with duration T_0 much longer than the carrier lifetime τ_c. This situation can be realized for even 10-ns pulses for SOI waveguides with a small mode area because their carrier lifetime is often reduced to close to 1 ns [41]. In the limit $T_0 \gg \tau_c$, the functions $g(z,\tau)$ and $w(z,\tau)$ are reduced to

$$\begin{aligned} g(z,\tau) &\approx 2\chi a_0^2(z,\tau)\psi(z,\tau,\tau)\tau_c \\ w(z,\tau) &\approx \varphi(z,\tau)\tau_c. \end{aligned}$$

It is easy to see that free carriers affect the pulse in a symmetrical fashion in this limit.

In the opposite limit of relatively short pulses ($T_0 \ll \tau_c$), we obtain the following order-of-magnitude estimates:

$$\begin{aligned} g(z,\tau) &\lesssim 2\chi a_0^2(z,\tau)\times\begin{cases} T_0\psi(z,\tau,\tau), & \tau<0 \\ \Psi\exp(-\tau/\tau_c), & \tau>0 \end{cases} \\ w(z,\tau) &\lesssim \begin{cases} T_0\varphi(z,\tau), & \tau<0 \\ \Phi\exp(-\tau/\tau_c), & \tau>0 \end{cases} \end{aligned}$$

where Ψ and Φ are total areas under the curves $\psi(z,\tau,\tau')$ and $\varphi(z,\tau)$, respectively

$$\Psi = \int_{-\infty}^{+\infty}\psi(z,\tau,\tau')\,d\tau', \quad \Phi = \int_{-\infty}^{+\infty}\varphi(z,\tau)\,d\tau.$$

This form can be understood by noting that the rise and decay of $g(z,\tau)$ and $w(z,\tau)$ occur on substantially different time scales. Physically speaking, the front end of the pulse is mainly affected by relatively fast TPA and Kerr nonlinearities (because free carriers are yet to be created), while the trailing edge is exposed to relatively slow free-carrier effects (FCA and FCD). In the case of ultrashort pulses, the entire pulse generates free carriers but it is barely affected by them. In contrast, in the case of long pulses, carriers are created so early that even the leading edge is affected by them. Numerical simulations [38], [18] lead to similar conclusions even in cases in which our analytic solution is inapplicable.

B. Approximation Applicable to Short SOI Waveguides and/or Moderate Input Pulse Powers

Despite all the simplifying assumptions, expressions (14) and (15) are still not easy to comprehend since they contain lengthy functions ψ and w. These equations can be simplified further in the case of relatively short waveguides and/or moderate input intensities. Without loss of generality, we write the amplitude of the input pulse in the form

$$u(\tau) = u_0 U(\tau)$$

where $u_0 = \sqrt{I_0}$ characterizes the amplitude of the electric field $(E_0 = \varpi u_0)$, I_0 is the input peak intensity, and $U(\tau)$ is the time envelope with $\max[U(\tau)] = 1$. Let us suppose that the waveguide length L obey the inequality

$$L^{-1} \gg \max\left(\alpha, \beta u_0^2\right). \tag{16}$$

This is equivalent to the assumption that TPA has little or no influence on the pulse envelope. In view of condition (16), we obtain from (9), (14), and (15) the following approximations:

$$a \approx u[1 - u^2(\beta + \chi\vartheta_2)z] \tag{17}$$
$$\phi \approx (\gamma u^2 - \xi\vartheta_4)z \tag{18}$$

where we have introduced a new function

$$\vartheta_k(\tau) = \int_{-\infty}^{\tau} \mathrm{e}^{-(\tau-\tau')/\tau_c} u^k(\tau')\, d\tau'$$

and k is an integer. Equations (17) and (18) show that, in the case under consideration, both the amplitude and the phase of the pulse vary linearly with the propagation distance. The presence of ϑ_2 and ϑ_4 in these equations clearly displays the cumulative nature of the free-carrier effects, FCA and FCD. The phase (18) shows that sign of the nonlinear phase shift is determined by the relative importance of the FCD and the Kerr effects and strongly depends on time and input peak intensity. This behavior is a consequence of the fact that an increase in the density of free carriers always lowers the refractive index of the optical mode.

V. Analysis of the Solution Applicability and Numerical Examples

It is instructive to look at the validity of our solution against direct numerical integration of the relevant equations that govern the pulse dynamics through SOI waveguides. We mainly concentrate on the case in which the influence of free carriers on the pulse amplitude is not too large. This amounts to the condition

$$\eta \equiv \max[g(z,\tau)] \ll 1. \tag{19}$$

We incorporate this relation together with all the validity conditions for (1) and (2) to determine the applicability domain of (6), (9), (11), and (12). In particular, we allow the waveguide length L and the peak intensity I_0 of input pulses to vary over many orders of magnitude. For definiteness, we assume that the pulse has a Gaussian shape so that

$$U(\tau) = \exp(-\tau^2/T_0^2).$$

The applicability domain of our analytic solution is displayed in Fig. 1 for three pulse durations in the range of 0.1 to 10 ps. In each case, the solution is valid in the region bounded by the corresponding curve. The horizontal parts of the curves result from the condition $L \ll L_d$. One can see that the shorter the pulse, the larger the admissible intensities for a given waveguide length. This becomes intuitively obvious if we note that for a given input peak intensity, shorter pulses generate fewer free carriers compared with high power pulses. Note also that longer waveguide lengths reduce the intensity range over which

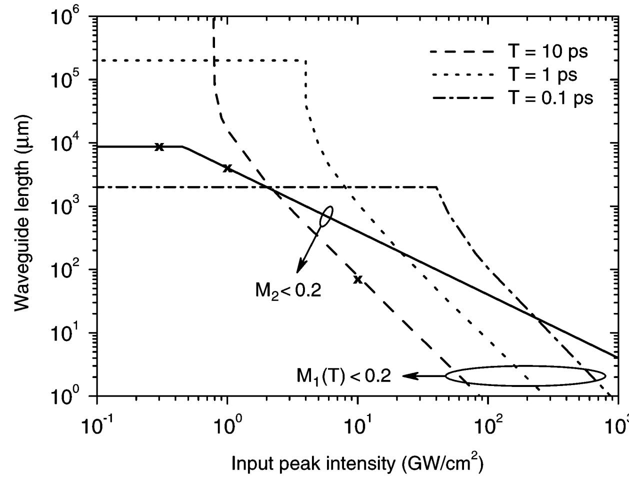

Fig. 1. Applicability domains of our analytic solution for three pulse widths. The area under each curve corresponds to the acceptable values of I_0 and L for that specific pulse width. $M_1(T) = \max(\eta, \delta, L/L_d)$. Solid curve shows the limitation due to condition (16) with $M_2 = \max(\alpha L, \beta u_0^2 L)$.

the solution applies because the number of TPA-induced free carriers scales quadratically with the intensity.

In addition to the validity conditions that must be satisfied, the applicability of (14) and (15) demands that the inequality (13) holds. However, this does not lead to any further restriction of the applicability domains presented in Fig. 1 since $\delta \le \eta$ irrespective of the pulse shape. Thus, in all cases of interest (14) and (15) can be used in place of (11) and (12). Note that, even when the condition (19) does not hold, (14) and (15) can be used to provide upper estimates of the free-carrier effects when $\eta < 1$.

The applicability of (17) and (18) requires the simultaneous fulfilment of the conditions (16), (13), and (19) as well as the validity conditions for (1) and (2). The solid line in Fig. 1 represents the condition (16). As an illustration, Fig. 2 shows changes in the nonlinear phase shift as a function of time for three specific values of I_0 and L at the edge of the applicability domain for 10-ps Gaussian pulses. Dashed lines show for comparison the phase shifts expected in the absence of the free-carrier effects. The free-carrier effects are relatively minor for less intense pulses (case marked pulse 3). SPM-induced spectral broadening in this case will lead to a nearly symmetric pulse spectrum. In contrast, the free-carrier effects become increasingly important for pulses with high peak intensities (pulses 1 and 2), and we would expect a highly asymmetric pulse spectrum in this case. As an example, the solid curve in Fig. 3 shows the spectrum of 10-ps pulse with $I_0 = 10$ GW/cm^2. In agreement with experiments and numerical modeling, FCA and FCD broaden the pulse spectra and shift it toward the shorter wavelengths.

A similar analysis can be carried out for pulses with other temporal envelopes, e.g., $U(\tau) = \mathrm{sech}(t/T_0)$ or $U(\tau) = \exp(-|t|/T_0)$. Our estimates show that the preceding conclusions remain qualitatively valid, even though the domain of the solution applicability becomes slightly different.

As a closing note, we stress that, even though our emphasis was on studying the dynamics of a single pulse propagating

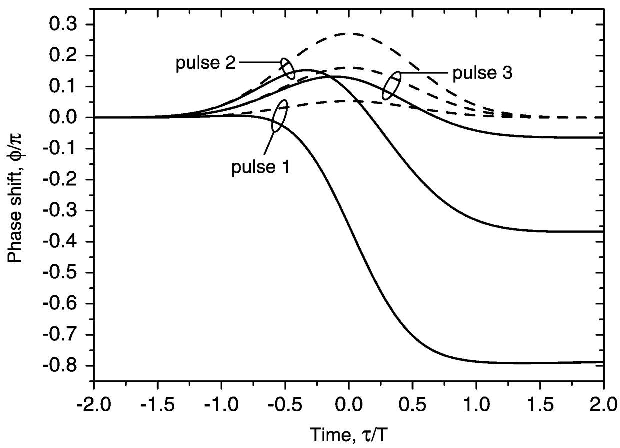

Fig. 2. Nonlinear phase shifts (solid curves) obtained from (15) for three boundary points shown by crosses in Fig. 1. The values of I_0 and L are 10 GW/cm^2 and 70 μm for pulse 1; 1 GW/cm^2 and 4 mm for pulse 2; and 0.3 GW/cm^2 and 8 mm for pulse 3. Dashed curves show phase shifts expected in the absence of FCD. Pulse duration is 10 ps in all three cases.

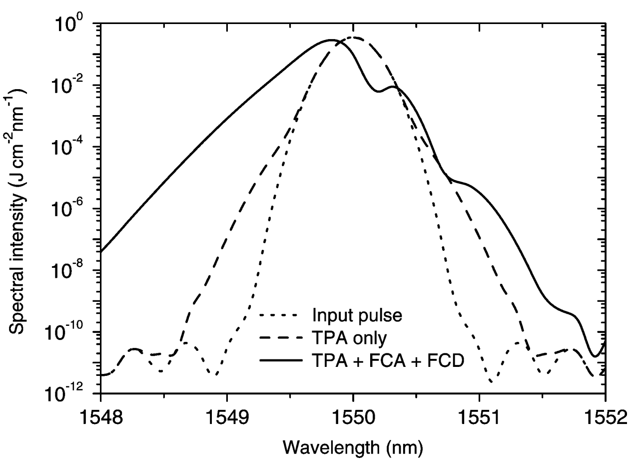

Fig. 3. Output spectrum of a 10-ps input pulse (solid curve) broadened by FCA and FCD effects when $L = 70\ \mu$m and $I_0 = 10$ GW/cm^2. The dashed curve includes the Kerr and TPA effects but ignores the impact of free carriers. Dotted curve shows for comparison the input pulse spectrum.

through a SOI waveguide, our results can be easily extended to characterize the propagation of an arbitrary pulse train. Indeed, if the function $u(\tau)$ contains several peaks, it can be thought of as the envelope of a pulse train. In this case, applicability of our solution will also depend on the repetition rate of the pulses. However, when the pule-to-pulse spacing is comparable to the carrier lifetime, it may be necessary to account for a background carrier density because the carrier density does not decay to zero before the next pulse arrives.

VI. Conclusion

In this paper, we have presented an approximate analytical solution describing evolution of short optical pulses inside SOI waveguides in the presence of Kerr nonlinearity, two photon-absorption, FCA, and index changes induced by the FCD. The emphasis has been to understand the impact of free carriers on the amplitude and the phase of optical pulses. The main approximations that we make are that: 1) the waveguide is shorter than the dispersion length and 2) FCA-induced changes in the pulse shape are relatively small. We have examined in detail the applicability domain of our analytic approach and show that it can be useful under realistic practical conditions. We have discussed several special cases in which our analytical solution is simplified considerably. Numerical examples with realistic device parameters show that our analytical solution has the accuracy and applicability to reproduce all main features of pulse evolution that have been obtained previously only by means of high-precision numerical modeling. We believe that our results will be useful for design optimization of SOI waveguides and lay methodological foundation to construct analytical solutions for active SOI waveguides such as silicon amplifiers in future.

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