Optical similaritons in nonlinear waveguides

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We discover analytically an extensive family of optical similaritons, propagating inside graded-index nonlinear waveguide amplifiers. We show that there exists a one-to-one correspondence between these novel similaritons and standard solitons of the homogeneous nonlinear Schrödinger equation. We demonstrate that for certain inhomogeneity and gain profiles, the newly discovered similaritons turn into solitons over sufficiently long propagation distances. © 2007 Optical Society of America

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Until recently, shape-preserving optical beams and pulses have been chiefly studied in passive, homogeneous, nonlinear systems [1,2]. Lately, however, a powerful generalization of the inverse scattering technique [3–5] has been developed to identify integrable inhomogeneous nonlinear systems and analytically determine stable shape-preserving waves supported by such systems [3–6]. In general, the discovered shape-preserving optical waves are self-similar structures maintaining their identity upon interactions. This circumstance prompted coining of the term *similariton* for such waves [7]. To date, temporal optical similaritons have been studied in homogeneous fibers and fiber amplifiers [8–10] as well as in dispersion-managed fiber amplifiers with distributed gain or loss [11–14]. The similariton interactions have also been investigated, and the possibility of bound-state multisimilariton formation upon similariton collisions was demonstrated [15].

However, much less attention has been paid to studying the behavior of solitons and self-similar waves in nonlinear systems exhibiting *both* spatial inhomogeneity and gain or loss [16] at the same time. A subtle interplay between the linear gain or loss, diffraction, and the inhomogeneity on the one hand, and the nonlinearity of such systems on the other, can result in a rich variety of shape-preserving waves with interesting properties.

In this Letter, we elucidate generic properties of (1+1)D shape-preserving waves in such systems by focusing on a specific example of tapered, graded-index, nonlinear waveguide amplifiers. The refractive index for such a system is given by

$$n(x,z) = n_0 + n_1 F(z) x^2 + n_2 I(x,z), \qquad (1)$$

where the first two terms describe the linear contribution to the refractive index and the last intensity-dependent term represents the Kerr nonlinearity. We assume $n_1 > 0$, but the dimensionless profile function $F(z)$ can be negative or positive, depending on whether the graded-index medium acts as a focusing or defocusing linear lens. The Kerr parameter n_2 is positive for nonlinear self-focusing but becomes negative in a self-defocusing medium.

The paraxial wave evolution corresponding to $n(x,z)$ in Eq. (1) is governed by the inhomogeneous nonlinear Schrödinger equation (NLSE) of the form

$$i\frac{\partial U}{\partial Z} + \frac{1}{2}\frac{\partial^2 U}{\partial X^2} + F(Z)\frac{X^2}{2}U - \frac{i}{2}G(Z)U \pm |U|^2 U = 0, \qquad (2)$$

where + (−) corresponds to the case of self-focusing (self-defocusing). The dimensionless variables are $Z = z/L_D$, $X = x/w_0$, and $U = (k_0|n_2|L_D)^{1/2}u$, where $k_0 = 2\pi n_0/\lambda$ is the wavenumber at the input wavelength λ and $L_D = k_0 w_0^2$ is the diffraction length with $w_0 = (2k_0^2 n_1)^{-1/4}$. Further, $G(Z) = [g(z) - \alpha(z)]L_D$ is a dimensionless net gain, where g and α account for linear gain and loss, respectively; $G > 0\,(<0)$ implies a net gain (loss) of energy in the system.

The aim of this Letter is to demonstrate that there exists a simple transformation that reduces Eq. (2) to a standard, homogeneous NLSE, which is well known to be integrable by the inverse scattering technique [17]. As a result, we uncover a one-to-one correspondence between any soliton of a homogeneous NLSE and a stable self-similar wave of the inhomogeneous NLSE, the stability of the latter following from the stability of the former. We also show that, for certain gain and index tapering profiles, there exists a subclass of similaritons of Eq. (2) that turn into true solitons over sufficiently long propagation distances.

We start by seeking a solution to Eq. (2) of the form

$$U(X,Z) = A(Z)\Psi\left[\frac{X - X_c(Z)}{W(Z)}, \zeta(Z)\right]e^{i\Phi(X,Z)}, \qquad (3)$$

where $A(Z)$, $W(Z)$, and $X_c(Z)$ are the dimensionless amplitude, width, and guiding-center coordinate of the beam, respectively. We assume a quadratic ansatz for the global phase:

$$\Phi(X,Z) = C(Z)X^2/2 + B(Z)X + D(Z). \qquad (4)$$

Substituting Eqs. (3) and (4) into Eq. (2), we obtain a set of first-order differential equations for the parameters of transformation (3) such that the transformed field Ψ satisfies the standard, homogeneous NLSE:

$$i\frac{\partial \Psi}{\partial \zeta} + \frac{1}{2}\frac{\partial^2 \Psi}{\partial \chi^2} \pm |\Psi|^2\Psi = 0. \qquad (5)$$

Here the effective propagation distance ζ is given by

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$$\zeta(Z) = \zeta_0 + \int_0^Z \frac{\mathrm{d}S}{W^2(S)}, \tag{6}$$

and we have introduced the similarity variable χ as

$$\chi(X,Z) = [X - X_c(Z)]/W(Z). \tag{7}$$

The guiding-center position is given by the expression

$$X_c(Z) = (X_0 + B_0 Z)/W(Z), \tag{8}$$

where $B(0) \equiv B_0$ and we chose $W(0)=1$.

The final form of the transformation from Eq. (2) to Eq. (5) can be written as

$$U(X,Z) = \frac{1}{W(Z)} \Psi\left[\frac{X - (X_0 + B_0 Z)/W}{W}, \zeta(Z)\right] \times \exp\left[\frac{iX^2}{2W}\frac{\mathrm{d}W}{\mathrm{d}Z} + \frac{iB_0 X}{W} - \frac{iB_0^2}{2}\int_0^Z \frac{\mathrm{d}S}{W^2(S)}\right], \tag{9}$$

where the (dimensionless) similariton width $W(Z)$ obeys the following simple, second-order, differential equation

$$\mathrm{d}^2 W/\mathrm{d}Z^2 - F(Z) W = 0. \tag{10}$$

We emphasize that transformation (9) can be realized only if the dimensionless gain and similariton width are related as

$$G(Z) = -\mathrm{d}[\ln W(Z)]/\mathrm{d}Z. \tag{11}$$

Transformation (3), which represents a combination of gauge and similarity transformations, together with a generalized scaling with respect to the Z variable, reduces Eq. (2) to the exactly integrable homogeneous NLSE [17]. It is instructive to compare our approach with that previously employed in a related context and termed the generalized inverse scattering method [5]. One of the new features of our method is an explicit Galilean invariance of all novel similariton solutions in the transformed variables. Consequently, one can obtain a wider class of shape-preserving solutions than those discussed in Refs. [5,6] by admitting arbitrary relative soliton/similariton velocities as free parameters. Another novel degree of freedom is provided by the choice of B_0, which specifies the initial position of the phase-curvature center and determines the dynamics of the guiding-center similariton.

Bright and dark similaritons. As an example, it follows readily from the well-known single-soliton solutions of the homogeneous NLSE [17] that the intensity profile $I=|U|^2$ of the most general bright single similariton is given by

$$I_B(\chi,\zeta) = (a^2/W^2)\mathrm{sech}^2[a(\chi - v\zeta)], \tag{12}$$

where a and v are the soliton amplitude and velocity, the latter being an additional free parameter. In the same way, the intensity profile of the dark single similariton is found to be

$$I_D(\chi,\zeta) = (u_0^2/W^2)[\cos^2(\phi)\tanh^2(\Theta) + \sin^2(\phi)], \tag{13}$$

where u_0 is the background amplitude and ϕ governs the grayness and speed of the dark soliton. The soliton phase Θ is defined in terms of these two parameters as [18]

$$\Theta(\chi,\zeta) = u_0 \cos(\phi)[\chi - u_0 \zeta \sin(\phi)]. \tag{14}$$

To gain further insight into the dynamical behavior of the novel similaritons, we consider some specific cases. We begin by observing that Eq. (10) is formally identical to a wave equation governing the modes of an inhomogeneous planar waveguide with the refractive index profile given by the function $F(Z)$. It then follows from the theory of sech^2-profile waveguides [19] that the lowest-order mode of such a waveguide corresponds to

$$F(Z) = 1 - 2\,\mathrm{sech}^2(Z), \quad W(Z) = \mathrm{sech}(Z). \tag{15}$$

From the compatibility condition in Eq. (11), the required gain profile is found to be $G(Z)=\tanh(Z)$.

The gain, width, and tapering profiles are displayed in Fig. 1(a). It can be seen from the figure that the tapering function $F(Z)$ crosses zero near $Z=1$, implying that the linear inhomogeneity of the waveguide should change from focusing to defocusing type. The required normalized gain $G(Z)$ is zero initially and tends toward 1 asymptotically for large Z. Such a gain distribution can be realized, for example, in an erbium-doped waveguide by suitably adjusting the density of the dopants. The similariton width under such conditions decreases monotonically. Hence, solutions (12) and (13), with the width profile specified in Eq. (15), describe self-focusing bright and dark fundamental similaritons. Their evolution with Z is displayed in Fig. 2, using the values $B_0=0.3$, $v=0.3$, $a=u_0=1$, and $\phi=0$.

Similaritons and solitons. A very interesting subclass of new similaritons is obtained for the tapering profile $F(Z)=-2\,\mathrm{sech}^2(Z+Z_*)$, which leads to the following width profile:

$$WZ = \tanh(Z+Z_*)/\tanh(Z_*). \tag{16}$$

Here Z_* is a positive constant, characterizing the asymptotic value of the similariton width, $W_\infty = \tanh^{-1}(Z_*)$. The evolution of fundamental bright and dark similariton corresponding to $W(Z)$ in Eq. (16) is shown in Fig. 3, using $Z_*=0.5$. Other param-

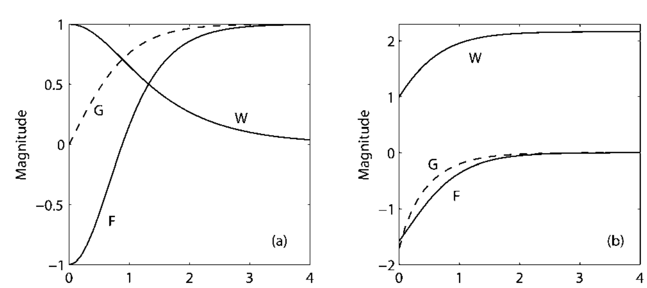

Fig. 1. Gain, width, and tapering profiles, plotted as functions of Z, for two specific choices of $F(Z)$ discussed in the text.

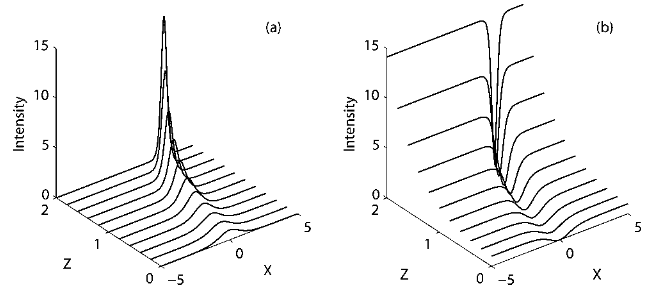

Fig. 2. Evolution of the fundamental (a) bright and (b) dark similaritons over two diffraction lengths under conditions of Fig. 1(a). Other parameters are specified in the text.

eters are the same as those used in Fig. 2. The corresponding tapering and gain profiles are shown in Fig. 1(b). Note that the gain G is negative (lossy medium) and tends toward zero for large Z. The width W, however, increases with Z, in spite of linear as well as nonlinear focusing, before attaining (asymptotically) a constant value, leading to soliton formation over just a few characteristic diffraction lengths. For example, in the case of $Z_{*}=0.5$ shown in Fig. 3, W differs from its asymptotic value by about 1% at $z=2L_D$, and this difference becomes less than 0.2% at a propagation distance of $3L_D$.

As a proof of the elastic character of the similariton interactions, we display in Fig. 4 the collision of two bright similaritons of equal amplitudes moving with equal but opposite velocities. For $Z<2$, the width and amplitude of each similariton is not constant, and its center moves according to Eq. (8). However, solitons form soon afterward as seen in the figure. Similar collision features occur for temporal similaritons [15].

In summary, we have found analytically a new class of exact self-similar waves supported by inhomogeneous gain media. The dynamical evolution of such waves is governed by a generalized inhomogeneous NLSE, which is shown to be exactly integrable by mapping it into the standard homogeneous NLSE. We have also discovered a subclass of the novel similaritons whose linear dimensions do not change in the long-propagation-distance limit, resulting in effectively static soliton solutions. The novel similaritons can serve as self-induced waveguides for guiding, amplifying, focusing and switching weak optical beams in inhomogeneous gain media. A graded-index, planar silica waveguide with erbium doping may be used for experimental realization of spatial similaritons discussed here. Another possibility is a suitably pumped dye-filled cell [20] in which density variations produce an index gradient.

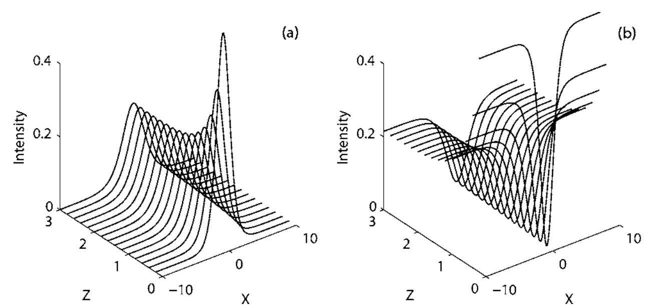

Fig. 3. Evolution of (a) bright and (b) dark similaritons over 3 diffraction lengths for the width and gain profiles of Fig. 1(b). Notice that solitons already form within 2 diffraction lengths.

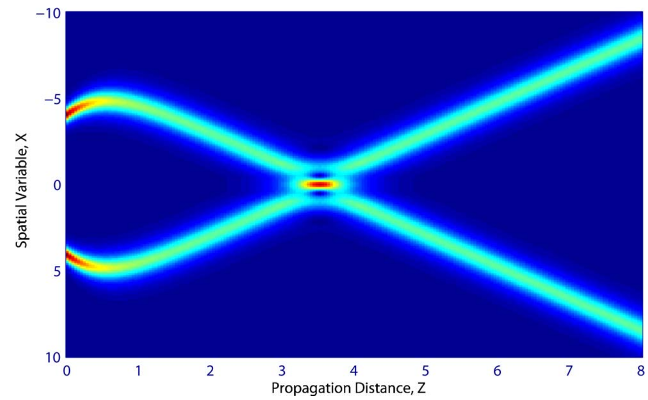

Fig. 4. (Color online) Collision of two bright similaritons, which turn into solitons for large Z.

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