Statistics of polarization-dependent gain in fiber-based Raman amplifiers

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We develop an analytic model for finding the statistics of polarization-dependent gain (PDG) in fiber-based Raman amplifiers. We use it to find an analytic form for the probability distribution of PDG and study how the mean PDG and the variance of PDG fluctuations depend on the PMD parameter. We show that mean PDG as well as PDG fluctuations are reduced by approximately a factor of 30 in the case of backward pumping.

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Raman amplification in optical fibers has become quite important for designing modern light-wave systems because of its potential for providing a relatively flat gain over a wide bandwidth.[1–3] Several experimental studies have shown not only that the gain of such amplifiers depends on the state of polarization of the input signal but also that this polarization-dependent gain (PDG) fluctuates over a wide range because of the random nature of polarization-mode dispersion (PMD) in optical fibers.[4,5] It is important to know the statistics of PDG, its relationship to the operating parameters of a Raman amplifier, and the conditions under which the PDG can be reduced to acceptable low levels. In this Letter we develop an analytic model for studying the statistics of PDG in fiber-based Raman amplifiers.

Introducing the Stokes vectors $\mathbf{P}$ and $\mathbf{S}$ for the pump and signal fields in the usual way[2] and using the general form of the nonlinear polarization for silica glass,[6] we obtain the following two vector equations:

$$\pm\frac{d\mathbf{P}}{dz} = -\alpha_p\mathbf{P} - \frac{\omega_p}{2\omega_s} g_R(P_0\mathbf{S} + S_0\mathbf{P}) + (\omega_p\boldsymbol{\beta} + \gamma_p\mathbf{W}_p^{\mathrm{NL}}) \times \mathbf{P}, \qquad (1)$$

$$\frac{d\mathbf{S}}{dz} = -\alpha_s\mathbf{S} + \frac{1}{2} g_R(S_0\mathbf{P} + P_0\mathbf{S}) + (\omega_s\boldsymbol{\beta} + \gamma_s\mathbf{W}_s^{\mathrm{NL}}) \times \mathbf{S}, \qquad (2)$$

where ω_p and ω_s are the carrier frequencies associated with the pump and signal waves, respectively, α_j and γ_j $(j = p, s)$ account for fiber losses and nonlinearities at these two frequencies, respectively, $P_0 = |\mathbf{P}|$ and $S_0 = |\mathbf{S}|$ represent the pump and signal powers, respectively, and g_R is the Raman-gain coefficient. We neglect the Raman gain for orthogonally polarized beams because of its small magnitude.[6,7] Birefringence vector $\boldsymbol{\beta}$ accounts for the PMD-induced rotation of the Stokes vectors on the Poincaré sphere, and the effects of nonlinear polarization rotation induced by self- and cross-phase modulations are governed by vectors $\mathbf{W}_p^{\mathrm{NL}} = 2/3(-2S_1, -2S_2, P_3)$ and $\mathbf{W}_s^{\mathrm{NL}} = 2/3(-2P_1, -2P_2, S_3)$. The choice of + and − signs in Eq. (1) depends on whether the pump beam copropagates or counterpropagates with the signal.

As a simplification, we assume that $P_0 \gg S_0$ and neglect pump depletion. Since the SRS process depends only on the relative orientation of $\mathbf{P}$ and $\mathbf{S}$, we choose to work in a rotating frame in which the pump polarization remains fixed. Pump equation (1) then contains only the loss term and can be easily integrated. Signal equation (2) takes the form

$$d\mathbf{S}/dz = -\alpha_s\mathbf{S} + \frac{1}{2} g_R(S_0\mathbf{P} + P_0\mathbf{S}) - \Omega_R\mathbf{b} \times \mathbf{S}, \qquad (3)$$

where $\Omega_R = \pm\omega_p - \omega_s$; the minus sign corresponds to backward pumping. Vector $\mathbf{b}$ is related to $\boldsymbol{\beta}$ by a rotation. Typically, the fiber length is much longer than the birefringence correlation length, and $\mathbf{b}$ can be modeled as a three-dimensional stochastic process with

$$\langle\mathbf{b}(z)\rangle = 0, \qquad \langle\mathbf{b}(z_1)\mathbf{b}(z_2)\rangle = \frac{1}{3} D_p^2 \overleftrightarrow{\mathbf{I}}\,\delta(z_2 - z_1), \qquad (4)$$

where $\overleftrightarrow{\mathbf{I}}$ is the second-order unit tensor and D_p is the PMD parameter.

The integration of Eq. (3) over fiber length L provides the amplifier gain [dB] $G = a \ln[S_0(L)/S_0(0)]$, where $a = 10/\ln 10 \approx 4.343$. The PDG is defined as $\Delta = G_{\max} - G_{\min}$ and is itself random because both the maximum and minimum values of G, which we obtain by changing the signal polarization, are random. Similarly to the case of polarization-dependent losses,[8,9] we introduce a PDG vector $\boldsymbol{\Delta}$ such that its magnitude gives the PDG value Δ and its direction corresponds to the direction of $\mathbf{S}(L)$ for which the gain is maximum. By use of Eq. (3), this vector is found to satisfy the following equation:

$$\frac{d\boldsymbol{\Delta}}{dz} = \frac{g_R\Delta}{2}\coth\left(\frac{\Delta}{2a}\right)[\mathbf{P} - (\mathbf{P}\cdot\hat{\Delta})\hat{\Delta}] + a g_R(\mathbf{P}\cdot\hat{\Delta})\hat{\Delta} - \Omega_R\mathbf{b} \times \boldsymbol{\Delta}. \qquad (5)$$

where $\hat{\Delta}$ is the unit vector in the direction of $\boldsymbol{\Delta}$. If the PDG is not too large, the first term in Eq. (5) can be Taylor expanded to the first order as $\Delta\coth(\Delta/2a) \approx 2a$. Equation (5) then reduces to linear Langevin equation

$$d\boldsymbol{\Delta}/dz \approx a g_R\mathbf{P} - \Omega_R\mathbf{b} \times \boldsymbol{\Delta}, \qquad (6)$$

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which can be solved easily. The solution is given by

$$\mathbf{\Delta}(z) = a g_R \overleftrightarrow{\mathbf{R}}(z) \int_0^z \overleftrightarrow{\mathbf{R}}^{-1}(z')\mathbf{P}(z')\mathrm{d}z', \tag{7}$$

where the PMD-induced rotation matrix $\overleftrightarrow{\mathbf{R}}(z)$ is obtained from $\mathrm{d}\overleftrightarrow{\mathbf{R}}/\mathrm{d}z = -\Omega_R \mathbf{b} \times \overleftrightarrow{\mathbf{R}}$. It can be shown that for fibers much longer than the birefringence correlation length the dynamics of $\mathbf{\Delta}$ corresponds to that of Brownian motion in three dimensions [see Ref. 10, where Eq. (7) is used for the PMD vector]. As a result, $\mathbf{\Delta}$ follows a three-dimensional Gaussian distribution.

The moments of $\mathbf{\Delta}$ can be obtained from Eq. (6) by use of a standard procedure.[11] It is appropriate to treat Eq. (6) in the Stratonovich sense. In the case of forward pumping, we obtain

$$\mathrm{d}\langle \mathbf{\Delta} \rangle/\mathrm{d}z = -\eta \langle \mathbf{\Delta} \rangle + a g_R P_{\mathrm{in}} e^{-\alpha_p z} \hat{P}, \tag{8}$$

$$\mathrm{d}\langle \mathbf{\Delta}^2 \rangle/\mathrm{d}z = 2 a g_R P_{\mathrm{in}} e^{-\alpha_p z} \hat{P} \cdot \langle \mathbf{\Delta} \rangle, \tag{9}$$

$$\mathrm{d}\overleftrightarrow{\mathbf{C}}/\mathrm{d}z = -3\eta \overleftrightarrow{\mathbf{C}} + \eta(\langle \Delta^2 \rangle \overleftrightarrow{\mathbf{I}} - \langle \mathbf{\Delta} \rangle \langle \mathbf{\Delta} \rangle), \tag{10}$$

where $\eta = 1/L_d = D_p^2 \Omega_R^2/3$, L_d is the PMD diffusion length, P_{in} is the power of the input pump polarized along the $\hat{P}$ direction, and $\overleftrightarrow{\mathbf{C}}$ is the covariance matrix defined as $\overleftrightarrow{\mathbf{C}} \equiv \langle \mathbf{\Delta}\mathbf{\Delta} \rangle - \langle \mathbf{\Delta} \rangle \langle \mathbf{\Delta} \rangle$. Equations (8) and (9) provide the following analytical results:

$$\langle \mathbf{\Delta} \rangle = \frac{a g_R P_{\mathrm{in}} \hat{P}}{\eta - \alpha_p} (1 - \alpha_p L_{\mathrm{eff}} - e^{-\eta L}), \tag{11}$$

$$\langle \Delta^2 \rangle = \frac{2(a g_R P_{\mathrm{in}})^2}{\eta^2 - {\alpha_p}^2} [(1 - \alpha_p L_{\mathrm{eff}}) e^{-\eta L} - 1 + L_{\mathrm{eff}}(\alpha_p + \eta)(1 - \alpha_p L_{\mathrm{eff}}/2)], \tag{12}$$

where L_{eff} is the effective fiber length defined as $L_{\mathrm{eff}} = [1 - \exp(-\alpha_p L)]/\alpha_p$.

An analytical expression of $\overleftrightarrow{\mathbf{C}}$ can also be obtained by integration of Eq. (10). It is convenient to choose $\hat{P}$ along an axis of the Stoke space, say $\hat{P} = \hat{e}_1$, so that $\overleftrightarrow{\mathbf{C}}$ is diagonalized. The probability density function of $\mathbf{\Delta}$ then can be written as

$$p(\mathbf{\Delta}) = \frac{(2\pi)^{-3/2}}{\sigma_\parallel \sigma_\perp^2} \times \exp\left[-\frac{(\Delta_1 - \Delta_0)^2}{2\sigma_\parallel^2} - \frac{(\Delta_2^2 + \Delta_3^2)}{2\sigma_\perp^2} \right], \tag{13}$$

where $\Delta_0 = |\langle \mathbf{\Delta} \rangle|$ and $\sigma_\parallel^2$ and $\sigma_\perp^2$ are the variances of the PDG vector in the parallel and perpendicular direction of $\hat{P}$, respectively. They can be obtained from Eq. (10) in analytical form. It turns out that $\sigma_\parallel < \sigma_\perp$ when PMD is small because the pump amplifies only the copolarized signal and prevents it from scattering. When $L_{\mathrm{eff}} \gg L_d$, $\langle \Delta^2 \rangle \gg \langle \mathbf{\Delta} \rangle^2$ so that $\sigma_\parallel \approx \sigma_\perp$.

Similarly to the case of the PMD vector, we need to find the distribution $p(\Delta)$, where $\Delta = |\mathbf{\Delta}|$, because that is what is measured experimentally. The distribution of Δ can be found from Eq. (13) after $\mathbf{\Delta}$ is written in spherical coordinates and after integrating over the two angles. The result is found to be

$$p(\Delta) = \frac{\Delta}{2\sigma_\parallel \sigma} \exp\{-[\Delta^2(r-1) - r\Delta_0^2]/2\sigma^2\} \times (\mathrm{erf}\{[\Delta(r-1) + r\Delta_0]/\sqrt{2}\,\sigma\} + \mathrm{erf}\{[\Delta(r-1) - r\Delta_0]/\sqrt{2}\,\sigma\}), \tag{14}$$

where $\sigma^2 = \sigma_\perp^2(r-1)$, $r = \sigma_\perp^2/\sigma_\parallel^2$, and erf is the error function. Figure 1 shows how $p(\Delta)$ changes with D_p for a 10-km-long fiber with losses of 0.273 and 0.2 dB/km for the pump and the signal, respectively. The pump power is 0.3 W, and the signal is located at the Raman-gain peak ($\Omega_R/2\pi = 13.2$ THz), where the gain coefficient $g_R = 0.61\ \mathrm{W}^{-1}/\mathrm{km}$.[1] In the limit $D_p \to 0$, $p(\Delta)$ becomes a delta function located at the maximum Raman gain, since no gain exists for the orthogonally polarized signal. As D_p increases, $p(\Delta)$ broadens quickly because PMD scatters the signal polarization randomly. When D_p is large enough that $L_{\mathrm{eff}} \gg L_d$, $p(\Delta)$ becomes Maxwellian, and its peak shifts to much smaller values. This is the behavior that was observed experimentally in the study reported in Ref. 4.

To compare our theory with experiments more quantitatively, we used the PDG distribution in Eq. (14) to calculated the mean value of PDG, $\langle \Delta \rangle$, and the rms value of PDG fluctuations, $\sigma_\Delta = (\langle \Delta^2 \rangle - \langle \Delta \rangle^2)^{1/2}$. Figures 2 and 3 show how these two quantities vary with the PMD parameter after both of them are normalized to the average gain $G_{\mathrm{av}} = a g_R P_{\mathrm{in}} L_{\mathrm{eff}}/2$ [dB] so that the curves are pump-power independent. The parameters values are the same as in Fig. 1. As expected, mean PDG decreases monotonically as D_p increases. Note, however, that $\langle \Delta \rangle$ can be as large as 30% of the average gain for $D_p = 0.05\ \mathrm{ps}/\sqrt{\mathrm{km}}$, and it decreases slowly with D_p after that, reaching a value of 8% for $D_p = 0.2\ \mathrm{ps}/\sqrt{\mathrm{km}}$. This is precisely what was

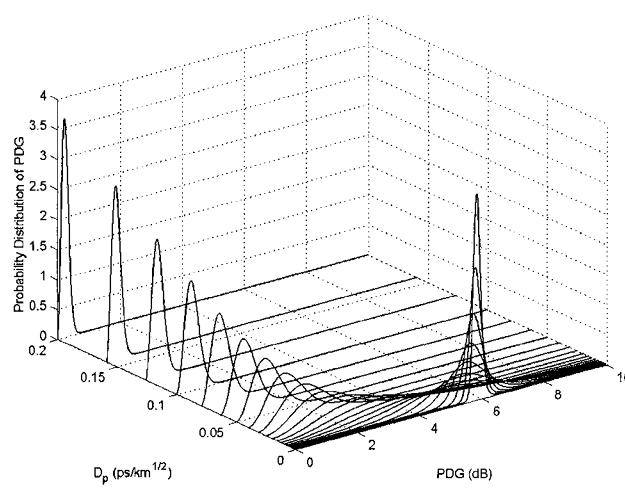

Fig. 1. Probability distribution of PDG as a function of D_p for a 10-km Raman amplifier pumped with 0.3 W of power in the forward direction.

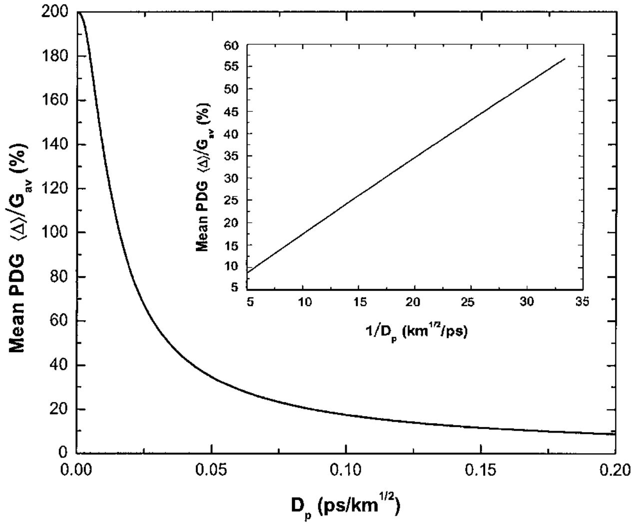

Fig. 2. Mean PDG as a function of PMD parameter, normalized to the average Raman gain. The inset shows the same data plotted as a function of $1/D_p$.

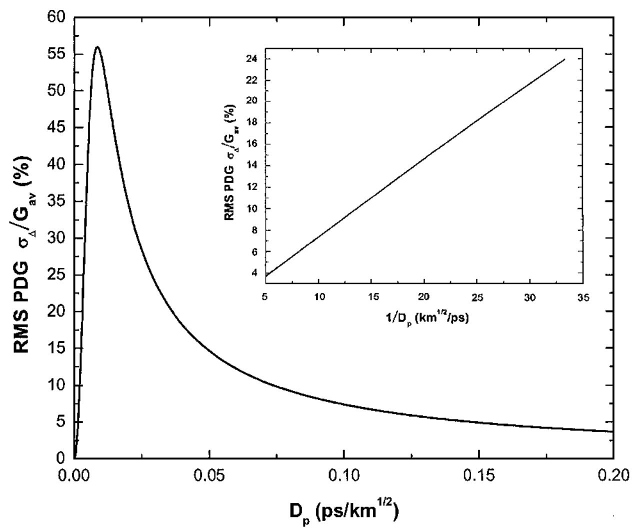

Fig. 3. Same as Fig. 2 expect that the rms value of PDG is plotted as a function of the PMD parameter.

reported in Ref. 4 through experiments and numerical simulations.

As can be seen in Fig. 3, the rms value of PDG fluctuations increases rapidly as D_p becomes nonzero, peaks at a value close to 56% for D_p near 0.01 ps/$\sqrt{\text{km}}$, and then begins to decrease. Again, fluctuations can exceed 7% of the average gain level even for $D_p = 0.1$ ps/$\sqrt{\text{km}}$. Both the mean PDG and the rms value of PDG fluctuations decrease with D_p inversely for $D_p > 0.03$ ps/$\sqrt{\text{km}}$ ($L_d < 0.5$ km for $\Omega_R/2\pi = 13.2$ THz). This can be seen from the insets in Figs. 2 and 3. The linear dependence agrees very well with the experimental results in Ref. 4.

The D_p^{-1} dependence of $\langle\Delta\rangle$ and σ_Δ can be deduced analytically from Eq. (14) in the limit $L_{\text{eff}} \gg L_d$. In this limit, the PDG distribution $p(\Delta)$ becomes approximately Maxwellian, and the average and rms values of PDG are given by

$$\langle\Delta\rangle \approx \frac{4ag_R P_{\text{in}}}{\sqrt{\pi}\, D_p \Omega_R}\,[L_{\text{eff}}(1 - \alpha_p L_{\text{eff}}/2)]^{1/2}. \quad (15)$$

$$\sigma_\Delta \approx \sqrt{(3\pi/8 - 1)}\,\langle\Delta\rangle \approx 0.422\langle\Delta\rangle. \quad (16)$$

The same equations hold in the case of backward pumping, except that $\Omega_R = \omega_p - \omega_s$ should be replaced with $\omega_p + \omega_s$. As a result, both $\langle\Delta\rangle$ and σ_Δ are reduced by a factor of $(\omega_p + \omega_s)/(\omega_p - \omega_s) \approx 2\omega_s/\Omega_R$ when backward pumping is used. Typically, this factor exceeds 30 in the 1.55-μm region. The peak value of the curve in Fig. 3 remains the same but its location shifts to smaller D_p by the same factor. The validity of Eqs. (9)–(16) depends on the approximation used for deriving Eq. (6). Using the second term in the Taylor expansion, we find that the validity condition is $\langle\Delta\rangle \ll \sqrt{12a} \approx 15$ dB. This requirement can be satisfied for most Raman amplifiers.

In conclusion, a general vector theory has been presented to describe the polarization-dependent gain in Raman amplifiers and to find its probability distribution as a function of the operating parameters of Raman amplifiers. The distribution is close to a Gaussian distribution when PMD is relatively small but becomes Maxwellian when the effective fiber length is much larger than the diffusion length. We were able to derive analytical expressions for the average and rms values of PDG. Our analytical predictions are in agreement with the experimental results and numerical simulations reported in Ref. 4.

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