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Non-integrability of equations governing pulse propagation in dispersion-managed optical fibers

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Abstract

We show that the equation governing pulse propagation in dispersion-managed optical fibers, as well as the reduced form of that equation, does not have conserved or periodically conserved quantities other than the mass, momentum, and (for the reduced equation only) the Hamiltonian. Implications of this result for the problem of four-wave mixing in collisions of pulses in optical telecommunication channels, are discussed. © 1999 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

The technique of dispersion management (DM), where one periodically compensates fiber dispersion by inserting in the transmission line segments of fiber (or Bragg gratings) with the opposite sign of dispersion, has become widely used in optical telecommunications. For a review of advantages which DM offers for the soliton data transmission format, see, e.g., Refs. [1,2]. The equation governing pulse propagation in dispersion-managed (DM) fibers is the nonlinear Schrödinger equation (NLS) where the dispersion coefficient is a piecewise-constant, periodic function of the propagation distance. This equation can be written in the following non-dimensional form:

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\big(d_0+D(z)\big)\frac{\partial^2 u}{\partial \tau^2}+u|u|^2=0\,. \tag{1}$$

Here we have used the fiber-optics convention, whereby the propagation distance, z, plays the role of the evolution variable, and the retarded time, τ, the role of the spatial variable. The dispersion coefficient in Eq. (1) is explicitly written as a sum of its average, constant part, d_0, and the periodic part, $D(z)$, whose average over the dispersion map period vanishes. Specifically, $D(z)$ takes on values D_1 and D_2 over the fiber segments of respective lengths L_1 and L_2, which compose the map, and

$$D_1L_1+D_2L_2=0\,. \tag{2}$$

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One can further normalize variables in Eq. (1) so as to have [3] the map period equal to unity, i.e.

$$L_1 + L_2 = 1\,, \tag{3}$$

and also

$$|D_1 L_1| = |D_2 L_2| = 1\,. \tag{4}$$

Without loss of generality, in what follows we take

$$D_1 L_1 = +1\,. \tag{4'}$$

In Eq. (1), we also assumed that the fiber is lossless.

From the applications standpoint, the most interesting is the case of the so-called strong DM, where both d_0 and $|u|^2$ are on the order of some small parameter ϵ. In that case, it was found numerically (see, e.g., Ref. [4]) that Eq. (1) can have long-living, pulse-like solutions for either sign of d_0, and even for $d_0 = 0$. Those solutions had a remarkably regular structure with oscillating tails, which was explained in Refs. [5,6] by representing the solution of Eq. (1) as a superposition of certain Hermite–Gaussian functions (see also Ref. [7]). Moreover, stationary solutions of the following integro-differential equation,

$$i\dot{a}(\omega) - \frac{1}{2} d_0 \omega^2 a(\omega) + \int d_{123}\, \delta_{12,3\omega}\, a(\omega_1)\, a(\omega_2)\, a^*(\omega_3)\, T_{123\omega} = 0\,, \tag{5}$$

to which Eq. (1) is reduced in the strong DM limit (i.e. for $d_0 \sim |u|^2 \sim \epsilon \ll 1$), were also found [8] to exhibit this regular structure. Eq. (5) was first derived in Ref. [9]; it was later re-derived in Refs. [8,10] by different techniques. In this equation, the overdot denotes $\partial/\partial z$, and

$$a(\omega, z) = \frac{1}{2\pi} \exp\left[\frac{i}{2} \omega^2 \int^z D(z')\, dz' \right] \times \int_{-\infty}^{\infty} d\tau\, \exp[-i\omega\tau]\, u(\tau, z)\,, \tag{6}$$

$$T_{123\omega} \equiv T(\omega_1, \omega_2, \omega_3, \omega) = \int_0^1 dz \exp\left[-\frac{i}{2} \left(\omega_1^2 + \omega_2^2 - \omega_3^2 - \omega^2 \right) \times \int^z D(z')\, dz' \right]. \tag{7}$$

Note that in Eqs. (6) and (7), the lower limit in the integral $\int^z D(z') dz'$ is not important, and in what follows we set it to be 0. In Eq. (5) and below we use the following shorthand notations:

$$\int d_{123} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1\, d\omega_2\, d\omega_3\,,$$

$$\delta_{12,3\omega} = \delta(\omega_1 + \omega_2 - \omega_3 - \omega)\,. \tag{8}$$

Given the numerical discovery in Ref. [4] of the highly regular structure of the pulse-like solutions of Eq. (1), one can ask whether that equation is integrable, or in some sense close to being integrable, at least for some values of its parameters $D(z)$ and d_0. A straightforward calculation using the Painlevé test answers the above question to the negative; the details are presented in Appendix A. However, one can still consider the following possibilities. First, even though Eq. (1) is non-integrable, yet the reduced equation, Eq. (5), can be integrable [11]. Second, integrability (at least in the (1 + 1)-dimensional case) usually implies the existence of infinitely many conserved quantities for the equation in question. Thus one can ask whether Eq. (1) has any 'quasi-conserved' quantities which would change periodically in z. Below we call such quantities 'periodically conserved'. It is natural to assume that the period with which such quantities could change equals that of the dispersion map. In this work, we use the method developed by Zakharov and Schulman [12,13] (see also Ref. [14] for a comprehensive review) and answer the above two questions to the negative.

The rest of this work is organized as follows. In Section 2 (Section 3) we prove that Eq. (5) (Eq. (1)) with $d_0 \neq 0$ can have no conserved (periodically conserved) quantities other than the mass, momentum, and the Hamiltonian. In fact, we show that for Eq. (1), even the Hamiltonian is *not* periodically conserved. However, it should be noted that Eq. (5) admits infinitely many functionals with a quadratic leading-order part, which are conserved up to the fourth order inclusively. A similar statement also holds for Eq. (1), with 'conserved' being replaced by 'periodically conserved', *and* provided that a certain condition on the coefficients $D(z)$ and d_0 is satisfied. In Sections 2 and 3 we also comment on the case $d_0 = 0$, which requires a slightly different analysis. In that case, we were also unable to find any additional conserved (or periodically conserved) quantities. Finally, in Section 4, we discuss the implications of our results for both the non-return-to-zero and soliton data transmission formats in optical telecommunications.

2. Conserved quantities of Eq. (5)

Following Ref. [13], we look for conserved quantities, I, of Eq. (5), in the following form:

$$I=\int d\omega f(\omega)|a_\omega|^2+\int (d\delta a)_{12,34} I^{(4)}_{1234} + \int (d\delta a)_{123,456} I^{(6)}_{123456}+\cdots, \tag{9}$$

where $a_\omega=a(\omega)$, $a_1=a(\omega_1)$, $I^{(4)}_{1234}=I^{(4)}(\omega_1,\omega_2,\omega_3,\omega_4)$, $(d\delta a)_{12,34}=d_{1234}\,\delta_{12,34}\,a_1 a_2 a_3^* a_4^*$, etc. We also note that due to our adoption of the fiber-optics notations in Eq. (1), the frequencies ω play the same role as the wave vectors k played in Ref. [13] and related works. The functions $I^{(4)}$ and $I^{(6)}$ satisfy the following symmetry relations with respect to their arguments:

$$I^{(4)}_{1234}=I^{(4)}_{2134}=I^{(4)}_{1243}=I^{(4)\,*}_{3412}, \tag{10}$$

$$I^{(6)}_{123456}=I^{(6)}_{P(123)P(456)}=I^{(6)\,*}_{456123}, \tag{11}$$

where $P(123)$ stands for any permutation of 1,2,3. Note that T_{1234}, defined in Eq. (7), also satisfies conditions (10).

The requirement that I be conserved means that $dI/dz=0$ at each successive order $O(a^n)$. At the order $O(a^2)$, this condition always holds. At the next order, $O(a^4)$, Eqs. (9) and (5) are used to obtain the condition

$$\int (d\delta a)_{12,34}\left(\frac{1}{2} f_{12,34} T_{1234} - d_0\, I^{(4)}_{1234}\Delta^{(4)}_{12,34}\right)=0, \tag{12}$$

where we have denoted

$$f_{12,34}=f(\omega_1)+f(\omega_2)-f(\omega_3)-f(\omega_4), \tag{13}$$

$$\Delta^{(4)}_{12,34}=\frac{1}{2}\left(\omega_1^2+\omega_2^2-\omega_3^2-\omega_4^2\right). \tag{14}$$

In deriving Eq. (12), we have used symmetry conditions (10) for T_{1234}. Eq. (12) requires that

$$I^{(4)}_{1234}=\frac{f_{12,34}}{2\,d_0\,\Delta^{(4)}_{12,34}} T_{1234}. \tag{15}$$

Then the $I^{(4)}$-term in expansion (9) is nonsingular provided that $f_{12,34}T_{1234}=0$ on the so-called [12] singular manifold

$$\omega_1+\omega_2-\omega_3-\omega_4=0, \quad \Delta^{(4)}_{12,34}=0. \tag{16}$$

It is known [15] that Eq. (16) has only two solutions:

$$(\omega_1=\omega_3,\quad \omega_2=\omega_4) \quad \text{and}$$
$$(\omega_2=\omega_3,\quad \omega_1=\omega_4), \tag{17}$$

both of which imply $f_{12,34}=0$. Hence a nonsingular fourth-order term in expansion (9) can always be found, and therefore we need to consider the next order, $O(a^6)$, of dI/dz. There we find a condition

$$\int (d\delta a)_{123,456}\Big(2\big[I^{(4)}_{(45-3)345}T_{126(12-6)} - I^{(4)}_{126(12-6)}T_{(45-3)345}\big] - d_0\,\Delta^{(6)}_{123,456}I^{(6)}_{123456}\Big)=0, \tag{18}$$

where $\Delta^{(6)}_{123,456}$ is defined similarly to $\Delta^{(4)}_{12,34}$ in Eq. (14), and $I^{(4)}_{(45-3)345}=I^{(4)}(\omega_4+\omega_5-\omega_3,\omega_3,\omega_4,\omega_5)$, etc. From Eq. (18) we find, for $d_0\neq 0$, that a nonsingular $I^{(6)}$ could only exist if one has

$$\int (d\delta a)_{123,456}\Big(I^{(4)}_{(45-3)345}T_{126(12-6)} - I^{(4)}_{126(12-6)}T_{(45-3)345}\Big)=0 \tag{19}$$

on the singular manifold

$$\omega_1+\omega_2+\omega_3-\omega_4-\omega_5-\omega_6=0, \quad \Delta^{(6)}_{123,456}=0. \tag{20}$$

We now show that Eqs. (19) and (20) cannot have a common solution, unless $f(\omega)=A\omega^2+B\omega+C$, where A,B,C are arbitrary constants. For the latter choice of $f(\omega)$, the conserved quantities are just the mass (also called the number of photons), momentum, and the Hamiltonian [12].

First, on manifold (20), one has $\Delta^{(4)}_{(45-3)345}+\Delta^{(4)}_{126(12-6)}=0$. Then one can use Eq. (15) to transform the l.h.s. of Eq. (19) into the form

$$\int (d\delta a)_{123,456}\frac{T_{(45-3)345}T_{126(12-6)}}{\Delta^{(4)}_{12,6(12-6)}} f_{123,456}, \tag{21}$$

where $f_{123,456}=f(\omega_1)+f(\omega_2)+f(\omega_3)-f(\omega_4)-f(\omega_5)-f(\omega_6)$. Next, it was shown in Ref. [15] that $f_{123,456}\neq 0$ (unless $f(\omega)=A\omega^2+B\omega+C$) on manifold (20), except on any of its submanifolds of the form (16). Then Eq. (19) could only hold if the first factor of the integrand in expression (21), properly symmetrized, vanishes on the manifold (20):

$$\sum_{\substack{ijk=P(123)\\ lmn=P(456)}} \frac{T_{ijn(ij-n)}T_{(lm-k)klm}}{\Delta^{(4)}_{ij,n(ij-n)}}=0\,. \tag{22}$$

In Eq. (22), indices i,j,k (l,m,n) take on values 1,2,3 (4,5,6) in any permutation; thus the sum on the l.h.s. contains 9 terms. To evaluate the l.h.s. of Eq. (22), we use the explicit expression for T_{1234}:

$$T_{1234}=\frac{\exp\left[-i\Delta^{(4)}_{12,34}\right]-1}{(-i)\,\Delta^{(4)}_{12,34}}\,, \tag{23}$$

found from Eqs. (7) and (2)–(4′), and also the following parametrization of manifold (20) [15]:

$$\omega_1=P+R\left(u+\frac{1}{u}-\frac{1}{v}+3v\right), \tag{24a}$$

$$\omega_2=P+R\left(u+\frac{1}{u}+\frac{1}{v}-3v\right), \tag{24b}$$

$$\omega_3=P-2R\left(u+\frac{1}{u}\right), \tag{24c}$$

$$\omega_4=P-2R\left(u-\frac{1}{u}\right), \tag{24d}$$

$$\omega_5=P+R\left(u-\frac{1}{u}+\frac{1}{v}+3v\right), \tag{24e}$$

$$\omega_6=P+R\left(u-\frac{1}{u}-\frac{1}{v}-3v\right), \tag{24f}$$

where P,R and u,v are independent parameters. Finally, we substitute Eq. (23) and Eq. (24) into Eq. (22) and evaluate it using the Mathematica symbolic calculations package. As a result, we find that Eq. (22), and hence Eq. (19), do not hold on manifold (20). [Note: We also verified that for $T_{1234}\equiv 1$, i.e. for the integrable NLS case, Eq. (22) does hold on manifold (20).] Consequently, $I^{(6)}$ must have a line singularity, which invalidates expansion (9). Thus additional conserved quantities of the form (9) do not exist for Eq. (5).

The above consideration breaks down for $d_0=0$ (cf. Eq. (12)), unless one has $f(\omega)=A\omega+B$, with A,B being arbitrary constants. Since the latter choice for $f(\omega)$ with either A or B nonzero corresponds to the conservation of the mass and/or the momentum, in which we are not interested here, we have to set $f(\omega)=0$ to satisfy Eq. (12). Then at the order $O(a^6)$ we find the condition to be exactly of the form (19) where now $I^{(4)}$ has not yet been determined. The only nonzero solution to that equation which we have been able to find is the obvious one, i.e. $I^{(4)}_{1234}=T_{1234}$. This corresponds to the Hamiltonian of Eq. (5) to be conserved. Furthermore, if we set $I^{(4)}_{1234}=0$ and $I^{(6)}_{123456}\neq 0$, then at the order $O(a^8)$ we find the condition

$$\int (d\delta a)_{1234,5678}\Big(I^{(6)}_{(567-34)34567}T_{128(12-8)} \\ -I^{(6)}_{12378(123-78)}T_{(56-4)456}\Big)=0\,, \tag{25}$$

where $I^{(6)}_{(567-34)34567}=I^{(6)}((\omega_5+\omega_6+\omega_7)-(\omega_3+\omega_4),\omega_3,\omega_4,\omega_5,\omega_6,\omega_7)$, etc. We verified that the most plausible choice for $I^{(6)}$:

$$I^{(6)}_{123456}=\sum_{\substack{ijk=P(123)\\ lmn=P(456)}} T_{ijl(ij-l)}T_{(mn-k)kmn}\,, \tag{26}$$

which satisfies symmetry conditions (11), does not satisfy Eq. (25). Thus, we were unable to find any additional conserved quantities of Eq. (5) for $d_0=0$, either.

3. Periodically conserved quantities of Eq. (1)

First, we rewrite Eq. (1) in the form similar to that of Eq. (5):

$$i\dot a_\omega-\frac{1}{2}d(z)\,\omega^2a_\omega+\int d_{123}\,\delta_{12,3\omega}\,a_1a_2a_3^*=0\,, \tag{27}$$

where $d(z)=d_0+D(z)$ and $a_\omega=(1/(2\pi))\int_{-\infty}^{\infty}d\tau\exp[-i\omega\tau]\,u(\tau,z)$. We look for a periodically conserved quantity in the form (9), with $f(\omega)$, $I^{(4)}$, $I^{(6)}$, etc. being independent of z. In Appendix B, we show that the same results as will be obtained below, also hold when these coefficients are periodic functions of z. (The latter case includes, in particular, the Hamiltonian of Eq. (1).) Thus, taking $f(\omega)$, $I^{(4)}$, $I^{(6)}$, etc. as being z-independent does not seem to limit the generality of our results. We show below that Eq. (1) has no periodically conserved quantities of the form (9), apart from the mass and the momentum, which are, in fact, conserved exactly rather than periodically. In particular, we show, at the end of this section, that the Hamiltonian of Eq. (1) is *not* periodically conserved.

Proceeding along the lines of the preceding section, we find at the order $O(a^4)$ a condition of the form (12) with $T_{1234} \equiv 1$ and d_0 being replaced by $d(z)$. Then the requirement

$$\int_0^1 \frac{dI}{dz} dz = 0 \tag{28}$$

at that order yields the following equation for $I^{(4)}$:

$$\frac{1}{2} f_{12,34} \int_0^1 dz \exp\left[-i \Delta^{(4)}_{12,34} \int_0^z d(z')\, dz' \right] = i I^{(4)}_{1234} \left(\exp\left[-i d_0 \Delta^{(4)}_{12,34} \right] - 1 \right). \tag{29}$$

In deriving the last equation, we have used an approximation

$$a_\omega = c_\omega \exp\left[-\frac{i}{2} \omega^2 \int_0^z d(z')\, dz' \right], \tag{30}$$

with c_ω being independent of z *at this order*. The r.h.s. of Eq. (29) vanishes for $d_0 \Delta^{(4)}_{12,34} = 2\pi M$, where M is any integer. Hence in order for a nonsingular $I^{(4)}$ to exist, the l.h.s. of Eq. (29) must vanish on any manifold of the form

$$\omega_1 + \omega_2 - \omega_3 - \omega_4 = 0,$$
$$d_0 \Delta^{(4)}_{12,34} = 2\pi M, \quad M \in \mathbb{Z}. \tag{31}$$

For $M = 0$, this always holds, as discussed in Section 2, because $f_{12,34} = 0$. For $M \neq 0$, $f_{12,34} \neq 0$ in general. Moreover, it is easy to show that solutions to Eq. (31) always exist even for $M \neq 0$, such that, e.g., $(\omega_1 - \omega_2)^2 \geq \max(0{,}8\pi M/d_0)$. Thus we must require that the integral on the l.h.s. of Eq. (29) vanish. This yields the following relation between the parameters of the dispersion map:

$$L_1 + \frac{1}{d_0} = N, \tag{32}$$

where N is an arbitrary integer. In deriving Eq. (32) we have used Eq. (2)–(4′). Thus, when condition (32) is satisfied, a nonsingular $I^{(4)}$ can always be found, and we need to proceed to the next order.

Our consideration of the order $O(a^6)$ is similar to that in Section 2, with one exception. Namely, in addition to the term which has the form of the l.h.s. of Eq. (18) with $T_{1234} = 1$ and d_0 replaced by $d(z)$, one also gets a contribution from the term which occured at the previous order (i.e. the one of the form of the l.h.s. of Eq. (12). That lower-order term contributes to the order $O(a^6)$, because c_ω in Eq. (30), which was used in the derivation of Eq. (29), does depend on z through terms of order $O(a^2)$ and higher (cf. Eq. (27)). Calculation of such terms would be a delicate task, due to the occurrence of a logarithmically diverging phase [16], and we do not perform it here. Instead, we use the following simple trick. Let us denote

$$F(z) = \left(\frac{f_{12,34}}{2} - d(z)\, \Delta^{(4)}_{12,34} I^{(4)}_{1234} \right) \times \exp\left[-i \Delta^{(4)}_{12,34} \int_0^z d(z')\, dz' \right]. \tag{33}$$

Then the z-integral over one map period of the l.h.s. of the counterpart of Eq. (12) for Eq. (1) can be rewritten as follows:

$$\int_0^1 dz \int (d\delta c)_{12,34} F(z) = -\int_0^1 dz \int (d\delta)_{12,34} \frac{\partial (c_1 c_2 c_3^* c_4^*)}{\partial z} \int_0^z F(z')\, dz'. \tag{34}$$

Here we have used integration by parts and the fact that $\int_0^1 F(z) dz = 0$ (cf. Eq. (29)). Also, the notation $(d\delta c)_{12,34}$ is analoguous to the notation $(d\delta a)_{12,34}$, defined after Eq. (9). Now the term $\partial(c_1 c_2 c_3^* c_4^*)/\partial z$ can be computed using Eqs. (30) and (27). Adding the result to the term of the form of the l.h.s. of Eq. (18), as discussed above, we finally arrive at the condition at the order $O(a^6)$:

$$2\,\mathrm{Re} \int_0^1 dz \int (d\delta c)_{123,456} \times \left(f_{12,6(12-6)} \int_0^z \exp\left[-i \Delta^{(4)}_{12,6(12-6)} \times \int_0^{z'} d(z'')\, dz'' \right] dz' - 2i I^{(4)}_{(45-3)345} \right) \times \exp\left[-i \Delta^{(4)}_{(45-3)3,45} \int_0^z d(z')\, dz' \right] = -\int (d\delta c)_{123,456} I^{(6)}_{123456} \times \left(\exp\left[-i d_0 \Delta^{(6)}_{123,456} \right] - 1 \right). \tag{35}$$

Note that the r.h.s. of this equation is always real due to symmetry conditions (11).

Singularities of $I^{(6)}$ can occur on any of the following singular manifolds:

$$\omega_1+\omega_2+\omega_3-\omega_4-\omega_5-\omega_6=0,$$
$$d_0\,\Delta^{(6)}_{123,456}=2\pi M,\quad M\in\mathbb{Z}. \tag{36}$$

On these manifolds, the l.h.s. of Eq. (35) is transformed into the form:

$$\begin{aligned}&\mathrm{Im}\int(d\delta c)_{123,456}\,f_{123,456}\\&\times\left(-2\left(\frac{1}{d_0+D_1}-\frac{1}{d_0+D_2}\right)^2\right.\\&\times\frac{\sin\left[\frac{N}{2}d_0\,\Delta^{(4)}_{12,6(12-6)}\right]\sin\left[\frac{N-1}{2}d_0\,\Delta^{(4)}_{12,6(12-6)}\right]}{\sin\left[\frac{1}{2}d_0\,\Delta^{(4)}_{12,6(12-6)}\right]}\\&\left.+\delta_{M,0}\left(\frac{L_1}{d_0+D_1}+\frac{L_2}{d_0+D_2}\right)\frac{1}{\Delta^{(4)}_{12,6(12-6)}}\right),\end{aligned} \tag{37}$$

where the integer N was defined in Eq. (32), and $\delta_{M,0}=1$ for $M=0$ and $\delta_{M,0}=0$ otherwise. In deriving expression (37), we have also used Eq. (2)–(4′) and (29). Now it sufficies to show that expression (37) does not vanish for $M=0$ (i.e. on singular manifold (20)), provided that $f(\omega)\neq A\omega^2+B\omega+C$. In fact, the term with $\delta_{M,0}$ does vanish there, as noted in the paragraph following Eq. (24). However, the rest of the expression does not vanish on manifold (20). The corresponding calculations, which require symmetrization of that term as per Eq. (22) and then use of Eq. (24), are too cumbersome even for Mathematica. The easiest way to verify that the term in question does not vanish is to simply evaluate it numerically (still with Mathematica or a similar package) for some particular values of d_0 and N and at some particular point on manifold (20). Alternatively, one could expand it in the Taylor series for small ω and subsequently convince oneself that the result is indeed nonzero. We performed the verification using both ways. Thus we have shown that even with condition (32) satisfied, a nonsingular $I^{(6)}$ does not exist. Hence Eq. (1) does not have additional periodically conserved quantities for $d_0\neq 0$.

As in Section 2, the above analysis requires some modification for the case $d_0=0$. Since that modification is a straightforward combination of the calculations used in this and the previous sections, we do not give its details here, but simply state the final result. Namely, we were unable to find additional periodically conserved quantities in this case, either.

To conclude this section, we now show that the Hamiltonian of Eq. (1) is not periodically conserved. The quadratic part of the Hamiltonian is:

$$H=\int d\omega\left(\frac{1}{2}d(z)\,\omega^2\right)|a_\omega|^2+\cdots, \tag{38}$$

i.e. in terms of expansion (9), $f(\omega)=d(z)\omega^2/2$. As shown in Appendix B, the same results are obtained for $f(\omega)=\omega^2/2$, and hence we present the details of our analysis for the latter form of f. We first consider the case $d_0\neq 0$. If condition (32) is not satisfied, then the $I^{(4)}$ found from Eq. (29) is singular on any of the manifolds (31) with $M\neq 0$, and therefore the Hamiltonian is not conserved already at order $O(a^4)$. If condition (32) is satisfied, then a nonsingular $I^{(4)}$ does exist, and at the next order, $O(a^6)$, we arrive at expression (37) where $f_{123,456}=\Delta^{(6)}_{123,456}$. To guarantee existence of a nonsingular $I^{(6)}$, this expression must vanish on any of the manifolds (36). For $M=0$, it does indeed vanish due to the special form of $f(\omega)$. However, for $M\neq 0$, where $f_{123,456}\neq 0$, it does not vanish identically. We verified this by taking random points on the following submanifold of manifold (36):

$$\begin{aligned}&\omega_1=\omega_2=0, &&\omega_3=\frac{2\pi M}{d_0}+\left(\frac{2\pi M}{d_0}\right)^2\alpha^2+\frac{3}{4},\\&\omega_{4,5}=\frac{1}{2}\pm\frac{2\pi M}{d_0}\alpha, &&\omega_6=\frac{2\pi M}{d_0}+\left(\frac{2\pi M}{d_0}\right)^2\alpha^2-\frac{1}{4}\end{aligned} \tag{39}$$

(where α is a real parameter) and evaluating the symmetrized, as per Eq. (22), expression (37) for various values of N and M. Thus we showed that the corresponding $I^{(6)}$ is singular even for $f(\omega)=\omega^2/2$, and hence there exists no periodically conserved quantity of Eq. (1) whose leading-order part would coincide with that of the Hamiltonian. In particular, the Hamiltonian itself is not conserved.

In the case $d_0=0$ the analysis is modified as follows. First of all, it easy to see that $f(\omega)$ must be identically zero. Then the first nontrivial condition arises at the order $O(a^6)$:

$$\int_0^1 dz \int (d\delta c)_{123,456} \times \left(I^{(4)}_{(45-3)345} \exp\left[-i\Delta^{(4)}_{12,6(12-6)} \int_0^z d(z')\,dz' \right] - I^{(4)}_{126(12-6)} \exp\left[-i\Delta^{(4)}_{(45-3)3,45} \int_0^1 d(z')\,dz' \right] \right) = 0\,. \tag{40}$$

Its obvious nonzero solution is

$$I^{(4)}_{1234} = \frac{\exp\left[-i\Delta^{(4)}_{12,34}\right]-1}{(-i)\,\Delta^{(4)}_{12,34}}\,. \tag{41}$$

Note that at this order, we have obtained the same result as for Eq. (5) with $d_0=0$ (cf. Eq. (23)). However, at the next order we already find a difference from the case of Eq. (5). At that order, $O(a^8)$, we obtain the following condition:

$$\begin{aligned} &2\,\mathrm{Re} \int_0^1 dz \int (d\delta c)_{1234,5678} \Bigg(I^{(4)}_{(34-5)(567-34)67} \\ &\quad \times \int_0^z \exp\left[-i\Delta^{(4)}_{34,5(34-5)} \int_0^{z'} d(z'')\,dz'' \right] dz' \qquad (49) \\ &- I^{(4)}_{(567-34)47(56-3)} \\ &\quad \times \int_0^z \exp\left[-i\Delta^{(4)}_{(56-3)3,56} \int_0^{z'} d(z'')\,dz'' \right] dz' \Bigg) \\ &\quad \times \exp\left[-i\Delta^{(4)}_{12,8(12-8)} \int_0^z d(z')\,dz' \right] \\ &\quad = \left(\frac{1}{D_1} - \frac{1}{D_2} \right) \mathrm{Re} \int (d\delta c)_{1234,5678} \\ &\qquad \times \left(I^{(6)}_{(567-34)34567} \frac{\exp\left[-i\Delta^{(4)}_{12,8(12-8)}\right]-1}{\Delta^{(4)}_{12,8(12-8)}} \right), \end{aligned} \tag{42}$$

where $I^{(4)}$ is given by Eq. (41). For $\Delta^{(4)}_{12,8(12-8)} = 2\pi M$, where M is a nonzero integer, $I^{(6)}$ is singular, unless the l.h.s. vanishes there. Using the above condition for $\Delta^{(4)}_{12,8(12-8)}$, the l.h.s. of Eq. (42) is transformed into the form:

$$\begin{aligned} &2\left(\frac{1}{D_1^2} - \frac{1}{D_2^2} \right) \mathrm{Re} \int (d\delta c)_{1234,5678} \\ &\quad \times \left(\frac{I^{(4)}_{(567-34)47(56-3)} \left(\exp\left[-i\Delta^{(4)}_{(56-3)3,56}\right]-1 \right)}{\Delta^{(4)}_{(56-3)3,56} \left(\Delta^{(4)}_{(56-3)3,56} + 2\pi M \right)} \right. \\ &\quad \left. - \frac{I^{(4)}_{(34-5)(567-34)67} \left(\exp\left[-i\Delta^{(4)}_{34,5(34-5)}\right]-1 \right)}{\Delta^{(4)}_{34,5(34-5)} \left(\Delta^{(4)}_{34,5(34-5)} + 2\pi M \right)} \right)\,. \end{aligned} \tag{43}$$

Obviously, on the 6-dimensional manifold

$$\omega_1+\omega_2+\omega_3+\omega_4-\omega_5-\omega_6-\omega_7-\omega_8=0,$$
$$\Delta^{(4)}_{12,8(12-8)} = 2\pi M \neq 0\,, \tag{44}$$

one has enough degrees of freedom to make expression (43) nonzero, with the only exception being the case of a symmetric dispersion map when $D_1=-D_2$. Note that in the latter case, $I^{(6)}$ does not vanish identically, because expression (43) holds only on manifold (44). Therefore, even if there exists a periodically conserved quantity with $I^{(4)}$ given by Eq. (41) and $I^{(6)}$ found from Eq. (42), it must be different from the Hamiltonian of Eq. (1) with $d_0=0$ (note that the latter also has $f(\omega)\not\equiv 0$). We did not investigate the possibility of existence of such a quantity in the case of a symmetric map, because the required calculations at the order $O(a^{10})$ are extremely cumbersome.

4. Summary and discussion

The main results of this study are the following. First, we showed (in Appendix A) that the Painlevé test fails for Eq. (1), thus indicating that that equation is not integrable by the Inverse Scattering Transform. Second, we applied the method of Zakharov and Schulman to the reduced form of Eq. (1) in the strong DM limit, Eq. (5), and showed that it does not have conserved quantities other than the mass (number of photons), the momentum, and the Hamiltonian. Third, using the same method, we showed that Eq. (1) does not have periodically conserved quantities other than the mass and the momentum, which are conserved exactly rather than periodically. We note here that there was a slight difference between the calculations for Eq. (5) and those for Eq. (1). While the former calculations are just the straightforward application of the Zakharov–Schulman method to the specific equation, the latter ones required a certain modification, which was explained in the paragraph immediately preceding Eq. (33). Let us also mention that our results for the case $d_0 = 0$, indicating the absence of conserved (or periodically conserved) quantities for Eqs. (1) and (5), are not truly rigorous. Rather, we showed that for some very plausible form of $I^{(4)}$ and $I^{(6)}$, additional conserved (or periodically conserved) quantities do not exist. Establishing a rigorous result in that case remains an open problem.

We also showed that when condition (32) holds, then Eq. (1) has *infinitely many* quantities with quadratic leading-order part, which are periodically conserved up to the order $O(a^4)$ inclusively. This fact can be interpreted as ‘approximate integrability’ in those cases where one is justified to treat the nonlinear terms in Eq. (1) as a small perturbation. This is precisely what occurs in the theory of the non-return-to-zero (NRZ) data transmission in optical telecommunications. There, a logical ONE is represented by a rectangular pulse which occupies the entire bit slot. If there are two adjacent ONEs, the field between them does not go to zero, in contrast to what occurs in the soliton transmission format. Hence the name ‘NRZ’. An important quantity in optical telecommunications is the detected pulse energy (which we called ‘mass’ or ‘number of photons’ above). Thus, since an NRZ pulse is approximately 5 times broader than the soliton which would represent the same bit of information, then its power (i.e. $|u|^2$) is correspondingly lower. Therefore, one usually treats nonlinear terms in the evolution of NRZ pulses as a small perturbation.

One of the serious detrimental effects caused by collisions of NRZ pulses belonging to different frequency channels with frequencies ω_1 and ω_2 is four-wave mixing (FWM), i.e. creation of a relatively small field at frequency, say, $2\omega_1 - \omega_2$, through the nonlinearity. If in addition to the frequency matching condition,

$$\omega_1 + \omega_1 = \omega_2 + (2\omega_1 - \omega_2)\,, \tag{45a}$$

the propagation constants, $\beta_\omega = d(z)\omega^2/2$, of these four waves satisfy the relation

$$\frac{1}{2} d_0\left(\omega_1^2 + \omega_1^2 - \omega_2^2 - (2\omega_1 - \omega_2)^2\right) L_{\mathrm{map}} = 2\pi M, \quad M \in \mathbb{Z}\,, \tag{45b}$$

then the FWM field with frequency $(2\omega_1 - \omega_2)$ will grow linearly (on average) with the propagation distance [17]. Note that Eq. (45) coincide with Eq. (31) above. Experimentally, one observes [17,18] sharp peaks of the FWM field, whose locations are determined by Eq. (45b), as one varies the frequency separation $(\omega_1 - \omega_2)$ (note that the l.h.s. of Eq. (45b) equals $-d_0(\omega_1 - \omega_2)^2$). However, if parameters of the dispersion map satisfy condition (32), then we predict that (almost) no FWM will be observed for any frequency separation. This conclusion, which in our analysis is a consequence of the aforementioned ‘approximate integrability’ of Eq. (1), is confirmed by the direct calculation of the FWM field [19]. In fact, our condition (32) follows from Eq. (8) of Ref. [19], where one has to set $N = 2$ and $\alpha = 0$. Note also that the same conclusion regarding the absence (or, more precisely, strong suppression) of FWM in collisions of NRZ pulses should also hold in the strong DM regime irrespective of the exact values of d_0 and L_1, because the corresponding equation, Eq. (5), is also ‘approximately integrable’ up to the order $O(a^4)$.

Since condition (32) implies ‘integrability’ of Eq. (1) only at the lowest nontrivial order (i.e., at $O(a^4)$), we expect that it should not be relevant for the dynamics of DM solitons. Indeed, the latter, in contrast to NRZ pulses, are essentially nonlinear objects, for which all terms $O(a^n)$ in expansion (9) play equally important role. Thus, DM solitons should exhibit features of solitary waves in non-integrable systems. One such feature is the inelastic interaction of adjacent DM solitons (with the same central frequencies), which was reported in Ref. [20]. Our own numerical simulations of Eq. (1), performed with condition (32) being imposed on the coefficients $D(z)$ and d_0, showed that the interaction remains inelastic under that condition as well.

More interesting, however, is the problem of FWM in collisions of DM solitons with disparate frequencies. On one hand, the results of Ref. [21], where this problem was considered for the conventional (i.e. NLS) solitons, suggest that a treatment based on the same assumptions as in the NRZ case (cf. their Eq. (4)), except that the shape of the soliton is now explicitly accounted for, yields good agreement with numerical simulations. On the other hand, since the soliton width is considerably smaller than the distance between consecutive solitons in a communication channel, there is, in general, asymmetry between the beginning and the end of the collision [22] in systems with periodically varying dispersion. This asymmetry, and hence the size of the FWM field, depends on a number of parameters, such as the pulse width, frequency separation, average dispersion, and lengths L_1 and L_2 $(=(1-L_1))$ of the fiber sections. We numerically simulated the collision of two DM solitons of Eq. (1), having fixed all of these parameters (at, respectively, $T_0=\sqrt{1/2}$ (cf. Ref. [5]), $\omega_1-\omega_2=5.8\pi$, and $d_0=1/17.5$) but L_1, which we varied between 0 and 1. The dependence of the energy of the FWM field appeared to be both irregular and strong, with variations between the maximum and minum values being more than two orders of magnitude. This numerical observation, along with the above argument involving Ref. [21], indicate that calculation of the FWM field in collisions of DM solitons is an interesting open problem.

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Appendix A. Painlevé test for Eq. (1)

Following the approach outlined, e.g., in Ref. [23], we consider the following independent expansions for u and $v\equiv u^*$:

$$u=\phi^{\alpha}\sum_{j=0}^{\infty}u_j\phi^j,\quad v=\phi^{\beta}\sum_{j=0}^{\infty}v_j\phi^j, \tag{A1}$$

where

$$\phi=\tau-\psi(z),\quad u_j=u_j(z),\quad v_j=v_j(z). \tag{A2}$$

Coefficients α, β and functions u_j, v_j, and $\psi(z)$ are to be determined. Since the details of this approach can be found, e.g., in Ref. [23], here we only give final results obtained at each step of the calculations. Substituting Eq. (A1) into Eq. (1), we obtain the following condition at the leading order:

$$\alpha=\beta=-1,\quad u_0v_0=-d(z). \tag{A3}$$

Next, at the order $O(\phi^{j-3})$ we find that the resonances [23] are located at

$$j=-1,0,3,4. \tag{A4}$$

Note that the first two resonances correspond to the arbitrariness of $\psi(z)$ and u_0, respectively. Therefore, we have to consider all the conditions arising at each successive order from $O(\phi^{1-3})=O(\phi^{-2})$ to $O(\phi^{4-3})=O(\phi)$. At the orders $O(\phi^{-2})$ through $O(\phi^0)$ we find, respectively, the following conditions:

$$u_1=\frac{iu_0}{d(z)}\frac{d\psi}{dz},\quad v_1=-\frac{iv_0}{d(z)}\frac{d\psi}{dz}, \tag{A5}$$

$$u_2=\frac{1}{3(d(z))^2}\left(i\frac{d(u_0d(z))}{dz}-u_0\left(\frac{d\psi}{dz}\right)^2\right),$$
$$v_2=\frac{1}{3(d(z))^2}\left(-i\frac{d(v_0d(z))}{dz}-v_0\left(\frac{d\psi}{dz}\right)^2\right), \tag{A6}$$

$$u_3\ \text{arbitrary},\quad v_3=\frac{1}{u_0^2}\left(u_3d(z)+\frac{u_0}{d(z)}\frac{d^2\psi}{dz^2}\right). \tag{A7}$$

Finally, at the order $O(\phi)$ we find that the necessary condition for u_4 and v_4 to exist is

$$\frac{1}{(d(z))^2}\left(3u_0d(z)\frac{d^2\,d(z)}{dz^2}-3u_0\left(\frac{d\,d(z)}{dz}\right)^2+\frac{du_0}{dz}\frac{d\,(d(z))^2}{dz}\right)=0. \tag{A8}$$

Eqs. (A7) and (A8) were obtained with the Maple symbolic calculations package. Obviously, Eq. (A8) does not hold for the piecewise-constant dispersion coefficient $d(z)$. Thus Eq. (1) does not pass the Painlevé test.

Appendix B. Coefficients in expansion (9) for periodically conserved quantities

Here we show that all conclusions of Section 3 remain the same if instead of constant coefficients $f(\omega)$, $I^{(4)}$, $I^{(6)}$, etc., considered there, one takes those coefficients to be periodic functions of z. Naturally, we assume that the periodicity is that of the dispersion map.

Consider the quantity $\int_0^1 (dI/dz)\, dz$. Using Eq. (9), where now the coefficients are functions of z, we find:

$$\begin{aligned}\int_0^1 \frac{dI}{dz} dz &= \int d\omega\Big(f(\omega, z=1)|a_\omega(z=1)|^2 \\ &\quad - f(\omega, z=0)|a_\omega(z=0)|^2\Big) + \cdots \\ &= \int_0^1 dz \int d\omega\, f(\omega, z=0)\, \frac{d}{dz}|a_\omega|^2 + \cdots .\end{aligned} \tag{B1}$$

We have not written explicitly the terms proportional to $I^{(4)}$, $I^{(6)}$, etc., because they have exactly the same form as the term in Eq. (B1). We have also used the periodicity condition: $f(\omega, z=1) = f(\omega, z=0)$. Thus the statement formulated at the beginning of this Appendix is proved.

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