Self-focusing of chirped optical pulses in nonlinear dispersive media

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The spatiotemporal self-focusing of chirped optical pulses propagating in a nonlinear dispersive medium has been studied analytically and numerically. The analytic theory shows that the critical power for self-focusing occurring in a dispersive media changes quadratically with the chirp parameter in both two and three dimensions. It is found that the critical wave action depends on the sign of the total chirp parameter. Analytic results show that the effect of chirp is similar to that of beam ellipticity except that ellipticity always increases the critical wave action. Numerical simulations are used to study the effect of chirp and group-velocity dispersion on self-focusing. It is shown numerically and analytically that the self-focusing process can be controlled by changing the chirp parameter.

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I. INTRODUCTION

Since 1970 it has been known that optical pulses can self-focus in both space and time while propagating in a nonlinear medium [1]. Interest in spatiotemporal self-focusing of an optical pulse in a nonlinear dispersive medium has been rekindled recently [2–6]. For self-focusing of subpicosecond and femtosecond pulses in a medium with fast (electronic) nonlinearity, group-velocity dispersion (GVD) becomes important because different temporal slices no longer focus independently in contrast with the quasi-cw moving-foci model [7]. Temporal reshaping effects due to dispersion then become significant. Different qualitative behavior can occur depending on the sign of GVD, that is, depending on whether the medium has normal or anomalous dispersion. Silberberg [2] suggested the possible existence of a stable light bullet when a medium has anomalous dispersion. In their numerical simulations, Chernev and Petrov [3,4] and Rothenberg [5] found that the presence of normal GVD causes splitting of the original input pulse into two temporally separated pulses. The original pulse center does not self-focus, as the moving-foci model would predict. Rather dramatic pulse-compression effects associated with self-focusing occur in the temporally split pulses away from the pulse center [5]. It is well known that self-focusing in space can result in temporal compression of an optical pulse [8,9].

The previous work on spatiotemporal self-focusing [1–5] has assumed an unchirped optical pulse whose width is transform limited. In practice, the pulses emitted by mode-locked lasers are often chirped. Frequency chirp can also be imposed externally and is expected to influence the self-focusing process significantly. This paper is devoted to the study of spatiotemporal self-focusing of chirped optical pulses in a dispersive Kerr medium. The moment method [10] has been used to derive a virial theorem for the evolution of the center of mass of the pulse in a medium with anomalous GVD. It is found that the critical power for self-focusing increases quadratically with increasing chirp parameter no matter whether the pulses are down-chirped or up-chirped. Since analytic results for the case of normal GVD are difficult to obtain, we study spatiotemporal self-focusing of a chirped pulse by solving numerically the nonlinear Schrödinger equation (NSE).

II. THEORY

The spatiotemporal evolution of an optical pulse propagating in a dispersive medium with cubic nonlinearity is governed by a multidimensional NSE of the form [2]

$$i\frac{\partial A}{\partial z}+\frac{1}{2\beta_0}\nabla_\perp^2 A-\tfrac{1}{2}\beta_2\frac{\partial^2 A}{\partial t^2}+\frac{n_2\beta_0}{n_0}|A|^2A=0 , \qquad (1)$$

where A is the slowly varying envelope of the electric field associated with the optical pulse. The wave number $\beta_0=n_0k_0=n_0(\omega_0/c),\beta_2=\partial^2\beta/\partial\omega^2$ is the GVD parameter, n_0 is the linear part of the refractive index at the carrier frequency ω_0, n_2 is the nonlinear-index parameter responsible for self-focusing, and a retarded time frame traveling at the group velocity v_g is assumed. Absorption and higher-order effects such as third-order dispersion and self-steepening have been neglected. Equation (1) can be normalized to become [2]

$$i\frac{\partial A}{\partial z}+\tfrac{1}{2}\nabla_\perp^2 A-\tfrac{1}{2}\mathrm{sgn}(\beta_2)\frac{\partial^2 A}{\partial \tau^2}+|A|^2A=0 , \qquad (2)$$

where transverse coordinates x and y are normalized to the beam radius w_0, and the retarded time t is normalized such that $\tau=1/\sqrt{w_0\beta_2\beta_0}$. Since no analytic results could be obtained with $\mathrm{sgn}(\beta_2)=1$, we limit our analysis to the specific case of $\mathrm{sgn}(\beta_2)=-1$; that is, we only consider anomalous dispersion here. The difficulty in the case of $\mathrm{sgn}(\beta_2)=1$ is due to the fact that the diagonal elements in the dispersion tensor [11] have a different sign, and no functional analysis could be used to estimate the integrals in the final of the virial theorem. In a later section, we numerically study spatiotemporal self-focusing in a medium with positive GVD.

The spatiotemporal evolution of the wave amplitude can be described by the Lagrangian density [11,12]

$$L=\tfrac{1}{2}i\left[A^*\frac{\partial A}{\partial z}-A\frac{\partial A^*}{\partial z}\right]-\tfrac{1}{2}|\nabla_\perp A|^2-\frac{1}{2}\left|\frac{\partial A}{\partial \tau}\right|^2+\tfrac{1}{2}|A|^4 , \quad (3)$$

where A^* is the complex conjugate of A. The Euler-Lagrange equation is as follows:

$$\frac{\partial}{\partial z}\frac{\partial L}{\partial(\partial A/\partial z)}+\sum_{i=1,2,3}\frac{\partial}{\partial x_i}\frac{\partial L}{\partial(\partial A/\partial x_i)}-\frac{\partial L}{\partial A}=0 , \quad (4)$$

where $x_1=x$, $x_2=y$, and $x_3=\tau$.

The application of the Euler-Lagrange Eq. (4) to the Lagrangian density Eq. (3) generates the NSE given by Eq. (2). The symmetries and conservation laws of the NSE associated with Eq. (2) have been studied by many authors [13–15]. In this paper only three conservation laws are needed. They are gauge invariance, space translation, and time translation, corresponding to the conservations of wave action N [related to optical power in two dimensions (2D)], momentum P, and energy H, respectively. These conservation laws can be obtained either from Noether's theorem or directly from Eq. (2) [13–15]. The result is

$$N=\int|A|^2d^Dr , \quad (5)$$

$$P=\frac{1}{2i}\int(A^*\nabla A-A\nabla A^*)d^3r , \quad (6)$$

$$H=\int(\tfrac{1}{2}|\nabla A|^2-\tfrac{1}{2}|A|^4)d^3r , \quad (7)$$

where $r^2=\sum_{i=1}^{D}x_i^2$ and D is the dimension. In 3D, $r^2=x^2+y^2+\tau^2$. In the moment approach [110], the average value of a physical quantity $F(r)$ is defined by

$$\langle F\rangle=\frac{\int|A|^2Fd^Dr}{\int|A|^2d^Dr} . \quad (8)$$

The quantity of particular interest from the standpoint of self-focusing is the effective beam size (in both space and time dimensions) determined from

$$\langle\delta r^2\rangle=\langle r^2\rangle-\langle r\rangle^2 . \quad (9)$$

It is easy to show that

$$\frac{d^2}{d^2z}\langle\delta r^2\rangle=\frac{d^2}{d^2z}\langle r^2\rangle-2\frac{d}{dz}\langle r\rangle^2-2\langle r\rangle\frac{d^2}{d^2z}\langle\delta r\rangle . \quad (10)$$

By using a standard procedure [11], we obtain

$$\frac{d^2}{d^2z}\langle\delta r^2\rangle=4[H+R/2-P^2/(2N)]/N , \quad (11)$$

where R is given by

$$R=\int(1-D/2)|A|^4d^Dr , \quad (12)$$

and D is the dimensionality of beam propagation ($D=3$ when x, y, and τ are all included). If the right-hand side of Eq. (11) is negative, then the pulse width (both spatial and temporal) will decrease to zero in a finite distance, leading to beam collapse through catastrophic self-focusing. Since the wave action is a constant, the wave amplitude will become infinite when $\langle\delta r^2\rangle=0$. Therefore, a sufficient condition for self-focusing is that the right-hand side of Eq. (11) be negative. Since self-focusing depends strongly on the dimensionality, we consider two- (2D) and three-dimensional (3D) cases separately.

III. THE 2D SELF-FOCUSING OF A CHIRPED PULSE

If the input profile is assumed to be Gaussian in time and space, and the pulse is assumed to be chirped both spatially and temporally,

$$A(x,\tau,z=0)=A_0\exp\left[-\frac{x^2}{2\rho_1^2}(1+iC_1)\right]\times\exp\left[-\frac{\tau^2}{2\rho_2^2}(1+iC_2)\right] , \quad (13)$$

where A_0 is the peak amplitude of the input pulse; ρ_1 and ρ_2 are the initial spatial and temporal widths, respectively; and C_1 and C_2 are the spatial and temporal chirp parameters. It is easy to see that the effect of chirp is the introduction of a phase curvature, which corresponds to the lens effect. There is some analogy between the spatial and temporal chirps since the roles of transverse and temporal coordinates are symmetrical to each other. In space, a positive chirp ($C_1>0$) corresponds to a positive lens; however, the temporal lens effect depends on the dispersion of the nonlinear medium. A positive chirp (or up-chirp) in the time domain introduces a positive (negative) temporal lens effect when the medium is of anomalous (normal) dispersion. Since we are considering anomalous dispersion in this section, the roles of x and t are exactly the same; hence, the results can also be applied to the 2D self-focusing in space. Equation (13) describes a beam profile with both astigmatism ($C_1\neq C_2$) and ellipticity ($\rho_1\neq\rho_2$). As far as the intensity distribution is concerned, there is some relationship between astigmatism and ellipticity; however, the effects of astigmatism and ellipticity on self-focusing are quite different. This difference turns out to be important in understanding the self-focusing of a light beam with the initial convergence or divergence. We will return to this point later in this paper.

Before obtaining the exact solution of Eq. (11), we first discuss self-focusing by using a simple phenomenological model discussed in Ref. [16]. The basic idea of this model is that self-focusing occurs when the nonlinear refraction can balance the beam divergence caused by linear diffraction. The nonlinear refraction is estimated using the concept of total internal reflection. It should be mentioned that this model is valid for spatial self-focusing of an optical beam with a symmetric cross section. In this paper, the method is generalized to the spatiotemporal self-focusing of an optical pulse, or spatial self-focusing of an optical beam with an asymmetric cross section. Although the concept of total internal reflection is no longer valid in time domain, the method is still applicable due to the symmetry between the transverse space and time included in Eq. (2), when $\mathrm{sgn}(\beta_2)<0$. In other words, the time coordinate can be treated as another transverse coordinate. The linear evolution of a spatially and temporally chirped pulse can be obtained by the method of the Fourier transform [17]. The result is

$$A(x,\tau,z)=A_0\left\{\left[1+iz(1+iC_1)/\rho_1^2\right]\left[1+iz(1+iC_2)/\rho_2^2\right]\right\}^{-1/2} \times\exp\left\{-\frac{(1+iC_1)x^2}{2\rho_1^2\left[1+iz(1+iC_1)/\rho_1^2\right]}\right\}\exp\left\{-\frac{(1+iC_2)\tau^2}{2\rho_2^2\left[1+iz(1+iC_2)/\rho_2^2\right]}\right\}. \tag{14}$$

It is easy to show that the far-field divergence is given by

$$\theta_{\mathrm{c}m}=(1+C_m^2)\theta_{0m}\quad(m=1,2)\ , \tag{15}$$

where θ_c is the far-field divergence in space or time domain, θ_0 is the far-field divergence without chirp, and $\theta_{0m}=1/\rho_m, m=1,2$. Note that the beam divergence is independent of the sign of the chirp; both positive and negative chirps give the same beam divergence in the far field. Since the refractive index is higher where the intensity is higher, some optical rays could be trapped due to the total internal reflection for a Gaussian intensity profile. If the intensity is high enough that all rays bounded by the divergent angle given by Eq. (15) can be trapped, then self-focusing will occur. If the angle under which the light ray experiences total internal reflection for a given intensity distribution denoted by θ_{NL}, then the threshold conditions are [16]

$$\theta_{\mathrm{NL}}=\left[\frac{N}{N_0}\right]^{1/2}\left[\frac{1}{\rho_1\rho_2}\right]^{1/2}\geq\theta_{\mathrm{c}m}\ ,\quad m=1,2\ , \tag{16}$$

where $N=\int|A|^2dx\,d\tau$ is the wave action, and $N_0=\lambda^2c_0/64\pi^3n_2$ is the critical wave action when $\rho_1=\rho_2, C_{1,2}=0$. The addition of Eq. (16) for $m=1$ and 2 gives the critical wave action

$$\frac{N_c}{N_0}=\frac{1}{2}\left[\left[1+C_1^2\right]\frac{\rho_2}{\rho_1}+\left[1+C_2^2\right]\frac{\rho_1}{\rho_2}\right]. \tag{17}$$

Note that the threshold depends on both ellipticity and astigmatism. We will come back to this point after we obtain the exact analytic solutions for the threshold wave action.

We now consider the self-focusing phenomena by using the virial theorem [Eq. (11)]. In general, Eq. (11) cannot be solved analytically since R cannot be determined; however, $R=0$ in the 2D case [see Eq. (12)]. In this case the right-hand size of Eq. (11) is a constant since H, N, and P are constants of motion. As the occurrence of the self-focusing, or collapse, is defined by $\langle\delta r^2\rangle=0$, the self-focusing distance can be obtained from Eq. (11). The constants N, P, and H are calculated using Eq. (13) in Eqs. (5)–(7). The results are as follows:

$$N=\pi\rho_1\rho_2|A_0|^2\ , \tag{18}$$

$$P=0\ , \tag{19}$$

$$H=\frac{N}{2\rho_1\rho_2}\left[\tfrac{1}{2}(1+C_1^2)\frac{\rho_2}{\rho_1}+\tfrac{1}{2}(1+C_2^2)\frac{\rho_1}{\rho_2}-\frac{N}{N_0}\right], \tag{20}$$

where $N_0=2\pi$ is the critical wave action when $\rho_1=\rho_2, C_{1,2}=0$. Integration of Eq. (11) gives

$$\langle\delta r^2\rangle=\langle\delta r^2\rangle|_0+Bz+\frac{2H}{N}z^2\ , \tag{21}$$

where $\langle\delta r^2\rangle|_0$ characterizes the initial beam width, and B is the initial beam divergence $d\langle\delta r^2\rangle/dz|_{z=0}=-(C_1+C_2)$. The condition $\langle\delta r^2\rangle=0$ gives the self-focusing distances. The result is as follows:

$$\frac{z_f}{\rho_1\rho_2}=\frac{-C\mp\left[\frac{\bar\rho^2}{\rho_1\rho_2}\left[\frac{N}{N_0}-\tfrac{1}{2}(1+C_1^2)\frac{\rho_2}{\rho_1}-\tfrac{1}{2}(1+C_2^2)\frac{\rho_1}{\rho_2}\right]+C^2\right]^{1/2}}{\left[\frac{N}{N_0}-\tfrac{1}{2}(1+C_1^2)\frac{\rho_2}{\rho_1}-\tfrac{1}{2}(1+C_2^2)\frac{\rho_1}{\rho_2}\right]}, \tag{22}$$

where $\bar\rho=1/\sqrt{2}(\rho_1^2+\rho_2^2)^{1/2}$; $C=(C_1+C_2)/2$ is called the average chirp, while C_1+C_2 is called the total chirp. The threshold wave action can be obtained by requiring that z_f be a positive real number. It is found that the critical wave action depends on the sign of C and is given by

$$\frac{N_c}{N_0}=\begin{cases}\frac{1}{2}\left[(1+C_1^2)\frac{\rho_2}{\rho_1}+(1+C_2^2)\frac{\rho_1}{\rho^2}\right]-C^2\frac{\rho_1\rho_2}{\bar{\rho}^2}\;, & C>0 \quad (23)\\ \frac{1}{2}\left[(1+C_1^2)\frac{\rho_2}{\rho_1}+(1+C_2^2)\frac{\rho_1}{\rho_2}\right]\;, & C<0\;. \quad (24)\end{cases}$$

In order to gain more physical insight, consider 2D spatial self-focusing as an example. The discussion also applies to the 2D spatiotemporal self-focusing since x and t are symmetric mathematically. In 2D spatial self-focusing, N is just the critical power for self-focusing, $\rho_1\neq\rho_2$ means the elliptic beam profile, and $C_1\neq C_2$ means the astigmatism. Compared with Eq. (17), Eq. (23) shows that the critical power could be smaller when $C>0$. The physical meaning of $C>0$ is that the beam area contracts initially, so Eq. (23) means that a beam with initial convergence needs less power to have self-focusing compared to a divergent beam. Note that Eq. (24) agrees exactly with Eq. (17), which justifies our simple model in obtaining Eq. (17). In fact, we consider only the far-field divergence in obtaining Eq. (17), which means that only $C>0$ is included in the phenomenologic model. It is useful to consider several simple examples to understand the effects of astigmatism and ellipticity

A. Pure astigmatism ($\rho_1=\rho_2, C_1\neq C_2$).

Equations (23) and (24) become

$$\frac{N_c}{N_0}=\begin{cases}1+\frac{1}{4}(C_1-C_2)^2\;, & C>0 \quad (25)\\ 1+\frac{1}{2}(C_1^2+C_2^2)\;, & C<0\;. \quad (26)\end{cases}$$

Therefore, the critical power for both $C>0$ and $C<0$ is always higher than that of an aberrationless input beam. It should be pointed out that Eqs. (23) and (25) correspond to the situation when $H>0$, which means that the right-hand side of Eq. (11) is positive. Physically, a collapse for $H>0$ would correspond to a situation in which the contraction rate due to the positive total chirp ($C>0$) is sufficient to produce the collapse before the inhibiting effect of positive energy H can reverse the contraction. Equation (26) shows that a divergent beam requires more power to self-focus than a convergent beam by an amount quadratically proportional to the average chirp.

It has long been believed that there is a critical power at which self-focusing occurs, below which it cannot occur no matter how tightly the beam is focused at the entrance of the nonlinear medium [8]. This conclusion is based on following reasoning: At the focal plane, the transverse area decreases by a factor of f^2 when the beam is focused to reduce its width by a factor of f. As a result, the beam intensity increases by a factor of f^2 times for a given amount of beam power. However, the diffraction also increases f^2 times, which cancels the enhanced nonlinear refraction. Therefore, simply increasing the intensity cannot make the self-focusing happen. This can be seen from Eq. (25) by assuming $C_1=C_2$, which corresponds to a symmetric focal lens. The critical power equals that of an aberrationless beam; hence a focal lens has no effect on the critical power at all. When $C_1\neq C_2$, astigmatism is introduced; the critical power is also increased. As long as the beam has astigmatism, the critical power will be higher than that of an aberrationless beam. For example, a cylindrical lens will increase the critical power by an amount of $C_1^2/4$. In spatiotemporal self-focusing, a frequency chirp will also increase the critical power by an amount of $C_2^2/4$.

In practice, it is more important to know the minimum power required for a beam to collapse in a distance that equals the extension of the nonlinear medium. Although the critical power is independent of the initial convergence as in the case of $C_1=C_2>0$, the propagation process does depend on C. If we define the threshold power as the power that is sufficient for a beam to collapse in a finite distance 1, then this threshold power depends strongly on the initial convergence of the beam. For the sake of simplicity, we continue to study the special case of $\rho_1=\rho_2, C_1=C_2>0$. To determine the threshold power, let $z_f=L$ in Eq. (22); the result is as follows:

$$\frac{N_{\text{th}0}}{N_{\text{th}}}=\frac{1+L^2}{L^2+(LC-1)^2}\;, \quad (27)$$

where $N_{\text{th}0}$ is the threshold power required for a parallel beam ($C_1=C_2=0$), N_{th} is the threshold power for a convergent beam, and L is the normalized length of the nonlinear medium. When $LC=1$, $N_{\text{th}0}/N_{\text{th}}$ is usually a very large number, which means that the self-focusing occurs more easily if the beam has initial convergence. Typically, for a beam of width $w_0=1$ mm, a medium of length $l=10$ mm, and beam wavelength of 1 μm, the value of $N_{\text{th}0}/N_{\text{th}}=4\times10^3$. Thus, the threshold power necessary for a convergent beam to collapse in a finite distance is 4×10^3 times smaller than that of a parallel beam. Note that when $LC=1/f$, f is the focal length of the lens used to generate the spatial chirp $C_1=C_2=C$. $LC=1$ means that the focal length equals the length of the medium. It is easy to see from Eq. (27) that the condition $LC=1$ is optimal for the enhancement of self-focusing. For a weakly focused beam ($LC\ll1$), $N_{\text{th}0}/N_{\text{th}}$ approaches 1, which is obvious; while for a highly focused beam ($LC\gg1$), $N_{\text{th}0}/N_{\text{th}}$ approaches zero monotonically. This can be explained by the fact that the self-phase modulation (SPM), which is responsible for the nonlinear self-focusing, remains relatively small at the geometric focal point since SPM requires a longer distance to accumulate strong phase modulation. In other words, the propagation is dominated by linear propagation if the beam is highly focused. After the geometric point, the beam becomes divergent, and the self-focusing is depressed, rather than enhanced, by the initial focusing.

Although the above result also applies to spatiotemporal self-focusing, it is practically difficult to apply the same chirp on both space and time domains. On the other hand, $C_1 \neq C_2$ is more realistic in applications. A result similar to Eq. (27) can be obtained using the same calculation. Due to the presence of astigmatism, large enhancement, as in the case of $C_1 = C_2$, is no longer possible. A typical case is $C_1 = 0, C_2 > 0$ or $C_1 > 0, C_2 = 0$. The threshold wave action or power is given by

$$\frac{N_{\text{th0}}}{N_{\text{th}}} = \frac{1+L^2}{L^2+\frac{1}{2}+(2LC-1)^2/2} < 2 , \tag{28}$$

where C is the average chirp. Therefore, the largest reduction in the threshold is only a factor of 2 for a chirped pulse compared to an unchirped one. (The same is also true for the situation in which a cylindrical lens is used to generate a spatial chirp.)

In conclusion, both spatial and temporal chirps have great effect on self-focusing. The critical wave action or power depends on both the sign of the average chirp and the individual values of different chirps. A very special case is when both chirps are positive and equal to each other. It is found that the critical power is not dependent on the value of the chirp, while the threshold power, defined as the power necessary for a beam to collapse in a given distance, depends strongly on the chirp. The threshold power for a convergent beam could be 10^3 times smaller than that of a parallel beam. There is an optimal value of chirp for the largest enhancement of self-focusing.

B. Effect of ellipticity ($\rho_1 \neq \rho_2, C_1 = C_2 = 0$) on self-focusing

If there is no phase curvature in both space and time coordinates, then Eqs. (23) and (24) reduce to a simple equation

$$\frac{N_c}{N_0} = \frac{1}{2}\left[\frac{\rho_2}{\rho_1}+\frac{\rho_1}{\rho_2}\right] \geq 1 . \tag{29}$$

Hence, ellipticity always increases the critical wave action (or critical power in 2D spatial self-focusing). In practice, a widely used method to avoid spatial self-focusing is to use a cylindrical lens to make the beam elliptic and thus increase the threshold power; hence, the effect of astigmatism is similar to that of ellipticity. In fact, a beam with astigmatism will become elliptic during propagation. Finally, it should be pointed out that astigmatism is different from ellipticity in that ellipticity always increases the critical power, while astigmatism can either enhance or suppress self-focusing depending on the sign of the total chirp parameter.

IV. THE 3D SELF-FOCUSING IN MEDIUM WITH ANOMALOUS DISPERSION

The input profile can be obtained by generalizing Eq. (13) and is given by

$$A(x,y,\tau,z=0) = A_0 \exp\left[-\frac{x^2}{2\rho_1^2}(1+iC_1)\right] \times \exp\left[-\frac{y^2}{2\rho_2^2}(1+iC_2)\right] \times \exp\left[-\frac{\tau^2}{2\rho_3^2}(1+iC_3)\right] , \tag{30}$$

where x and y are the transverse coordinates; τ is the local time coordinate; C_1, C_2, and C_3 are the chirp parameters; ρ_1 and ρ_2 are the transverse beam width; and ρ_3 is the pulse width. Following the same method described previously, we can obtain a simple estimate for the critical wave action in the 3D case. Since the algebra is the same as that of the 2D case, we show only the results here. The critical wave action is given by

$$N_c = \tfrac{1}{3}\pi\sqrt{\pi}\rho_1\rho_2\rho_3 \sum_{i=1}^{3} \frac{1+C_i^2}{\rho_i^2} , \tag{31}$$

where N_c is the critical wave action for a spatially and temporally chirped pulse. In the 3D case, the wave action is the pulse energy. In three dimensions, R is not a constant; however, R can be evaluated in the same way as Cao and McKinstrie did in obtaining the threshold value for stable solitons in optical birefringent fibers [18]. The assumption for the validity of the method is that the pulse is localized in both space and time. Under this condition, we can use Cauchy inequality to obtain an upper bound on R, i.e.,

$$R < -N^2/(2V) , \tag{32}$$

where V is a constant representing the volume in which most of the pulse energy is located. Then the sufficient condition for collapse can be obtained by substituting Eq. (32) to the right-hand side of Eq. (11). If a Gaussian profile such as Eq. (30) is assumed, then the self-focusing distance and the critical wave action can be determined similarly. The self-focusing distance is given by

$$\frac{z_f}{\bar{\rho}^2} = \frac{-C \pm \left[\bar{\rho}^2\left[\frac{N}{V}+\frac{N}{2\sqrt{2\pi}\rho_1\rho_2\rho_3}-\sum_{i=1}^{3}\frac{(1+C_i^2)}{\rho_i^2}\right]+C^2\right]^{1/2}}{\frac{N}{V}+\frac{N}{2\sqrt{2\pi}\rho_1\rho_2\rho_3}-\sum_{i=1}^{3}\frac{(1+C_i^2)}{\rho_i^2}} , \tag{33}$$

where $C=(C_1+C_2+C_3)/2$, and $\bar{\rho}^2 = \frac{1}{2}(\rho_1^2+\rho_2^2+\rho_3^2)$. In 3D spatiotemporal self-focusing, the total chirp is defined as $C_1+C_2+C_3$. The critical wave actions are

$$N_c=\begin{cases}\dfrac{\sqrt{2}V}{V+\pi\sqrt{2\pi}\rho_1\rho_2\rho_3}\pi\sqrt{\pi}\rho_1\rho_2\rho_3\sum_{i=1}^{3}\dfrac{1+C_i^2}{\rho_i^2}-\dfrac{C^2}{4\bar{\rho}^2}\,, & C>0 \quad (34)\\[2ex] \dfrac{\sqrt{2}V}{V+\pi\sqrt{2\pi}\rho_1\rho_2\rho_3}\pi\sqrt{\pi}\rho_1\rho_2\rho_3\sum_{i=1}^{3}\dfrac{1+C_i^2}{\rho_i^2}\,, & C<0\,. \quad (35)\end{cases}$$

Again, the critical wave action depends on the sign of C. The effect of chirps has been explained in detail in the 2D case; the physics remains the same in 3D. However, it should be pointed out that the critical wave action in 3D depends not only on the ellipticity, but also on the individual length and time scales set by ρ_1, ρ_2, and ρ_3. The dependence on the spatial length scale is in agreement with well-known results [8]. Due to time-scale dependence, 3D self-focusing is more sensitive to the pulse profile. From Eqs. (34) and (35), one can see that a pulse with a steeper gradient in space or time has smaller critical energy. Hence in spatiotemporal self-focusing there is no absolute critical power or energy above which beam collapse will occur. In other words, beam collapse can occur for any value of pulse energy as long as the length scale (including time domain) is short enough. Of course, it should be kept in mind that the length scale must be large enough such that the slowly varying envelope approximation remains valid; otherwise, the governing equations used to obtain Eqs (33)–(35) are no longer valid. A direct consequence of Eqs. (34) and (35) is that even a long pulse may have 3D self-focusing. In the case of self-focusing of long pulses, such as the quasi-cw self-focusing, different temporal slices evolve differently; the leading and trailing edges have power less than the critical power, and thus become broadened due to diffraction. However, the center part of the pulse (assumed to be above critical power) experiences self-focusing, resulting in much steeper edges. If the edges are so steep that Eqs. (34) and (35) are satisfied, then the edges will collapse three dimensionally. Therefore, a long pulse can collapse three dimensionally if Eqs. (34) and (35) are satisfied. This phenomenon was studied numerically by Cao, McKinstrie, and Russell [19]. The critical wave action depends on the chirp in the same way as that of 2D case, i.e., it is dependent of the sign of the total chirp. Finally, note that Eq. (35) is almost the same as Eq. (31), except for a constant that determines the critical pulse energy when no spatial and temporal modulations are applied. Like the 2D case, both Eqs. (31) and (35) show the same dependence of critical energy on astigmatism and ellipticity. However, 3D spatiotemporal self-focusing is much more complicated than the 2D case, since both length scales and time scales are coming into play in 3D self-focusing.

V. NUMERICAL SIMULATIONS

It is interesting to check the theoretical results by computer simulations. Here we solve the 2D NSE using the split-step method [20] with a 128×128 grid. The computer code conserves energy exactly, which makes it useful in finding the critical threshold. As a demonstration, the following parameters are chosen: $\rho_1=\rho_2=1$, and $C_1=C_2=C=0$ or -1. From Eq. (24), the critical power (or wave action) $N_c=2\pi(1+C^2)$. Figure 1(a) shows the variation of the on-axis intensity during the propagation when $C=0$. The two curves correspond to the input power 8% above and below the critical value 2π. The simulation results agree very well with the theory. In fact, the agreement can be improved to within 1% if a larger numerical grid (256×256) is used. Figure 1(b) shows the results under conditions identical to those of Fig. 1(a) except $C=-1$ and the two curves correspond to input power 25% above and below the threshold. The theoretic threshold is 4π for $C=-1$, while the numerical

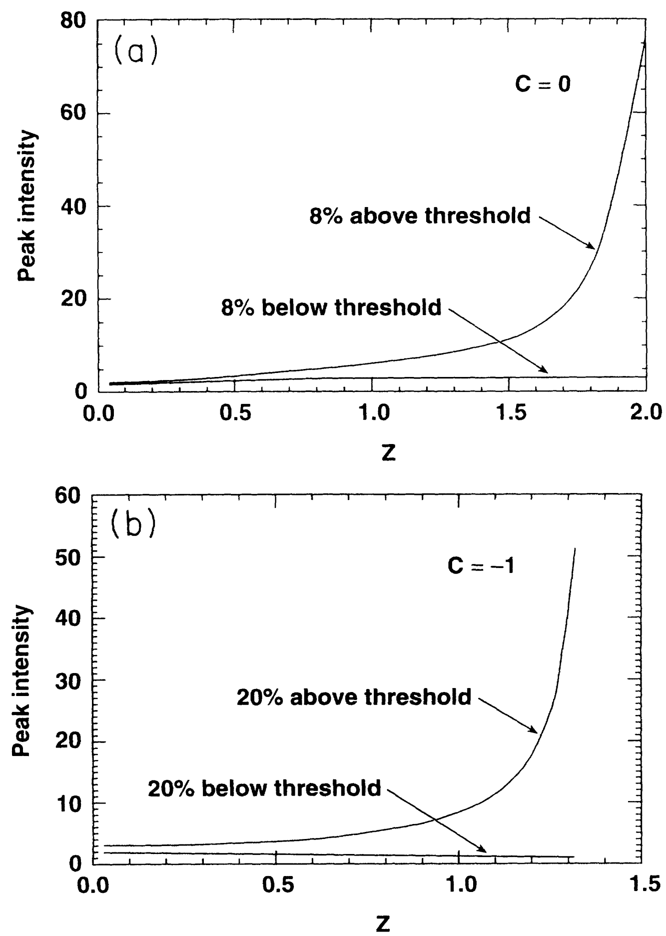

FIG. 1. On-axis intensity vs propagation distance for the critical power both above and below threshold for the case of $D=(2+1)$. (a) $C=0$. (b) $C=-1$. Z is in units of the Rayleigh range kw_0^2.

threshold is about 3.24π, the difference between them is 20%. This discrepancy is not surprising in view of the difference in the definition of the threshold between the theory and the simulations. The theoretical threshold corresponds to the situation in which the whole beam collapses into a single point, while the numerical threshold is measured by the peak intensity. It is well known that the collapse occurs before all the energy goes to the singular point. We have found that the agreement is quite good when $C<1$. For $C>1$, the agreement is only qualitative. The reason is that the required threshold power $N_c=2\pi(1+C^2)$ increases rapidly as C increases. Because the process of undoing the initial chirp is not uniform across the beam, it is possible for the center part to become chirp free and contain enough power to collapse. Therefore, the comparison between theory and simulation is expected to be only qualitative for $C>1$.

To study the effect of chirping on 3D spatiotemporal self-focusing, we solve Eq. (2) numerically again using the split-step method [20], but the beam is assumed to be cylindrically symmetric so that a fast Hankel transform [21,22] can be applied to the spatial coordinates, and the standard fast Fourier transform (FFT) has been used in time domain. Details about the numerical method can be found in Ref. [23]. A 128×128 grid has been used. The grid points in r (transverse coordinate) are not uniform (the smaller the radius r, the denser the grid points), a consequence of the method used for fast Hankel transformation [21]. The initial field is given by Eq. (30). First we consider the case when an optical pulse propagates in a nonlinear medium with normal GVD. It is known that in a linear medium with normal GVD, a pulse without initial phase modulations will become up-chirped ($C>0$) after propagating through the medium [16]. Therefore, an initially up-chirped pulse will broaden more rapidly when propagating in a linear medium, while a down-chirped pulse will be compressed initially and then broaden later, which is referred to as temporal focusing. Since the chirp will change the instantaneous power of a laser pulse, the character of spatiotemporal self-focusing will also change. As shown in this paper, both the critical power and the self-focusing distance are changed. We first simulate the 3D spatiotemporal self-focusing in a nonlinear medium with normal dispersion. Figure 2 shows the evolution of the on-axis intensity with initial parameters $A_0=5$, $\rho_1=\rho_2=\rho_3=1$, $C_1=C_2=0$, $C_3=-5$, $C_3=0$, and $C_3=5$, respectively. For an initially up-chirped ($C_3=5$) pulse, the temporal focusing is expected to enhance the self-focusing, while an initially down-chirped ($C_3=-5$) pulse will have the temporal defocusing effect, which will suppress the self-focusing (see Fig. 2). Figure 2 indicates no catastrophic collapse since only the on-axis intensities are shown. As pointed out in Refs. [3–5], the pulse can be split in time domain if the nonlinear medium has a positive GVD; hence the on-axis intensity will not always increase. Figure 3 illustrates the same self-focusing process but in a nonlinear medium with anomalous GVD. It is easy to see that a catastrophic self-focusing has occurred in this case. Since all the input parameters are the same as those used in Fig. 2, clearly the dispersion property of the nonlinear medium has a great effect on self-focusing. In other words, a positive GVD can suppress the self-focusing effect compared to the situation of a negative GVD. Also of importance is the fact that the self-focusing can be enhanced or suppressed by using chirped pulses for both normal and anomalous GVD. The self-focusing distance can also be controlled by changing the chirp parameters associated with the input pulses. The chirp introduced initially is expected to undo itself when approaching the self-focusing point.

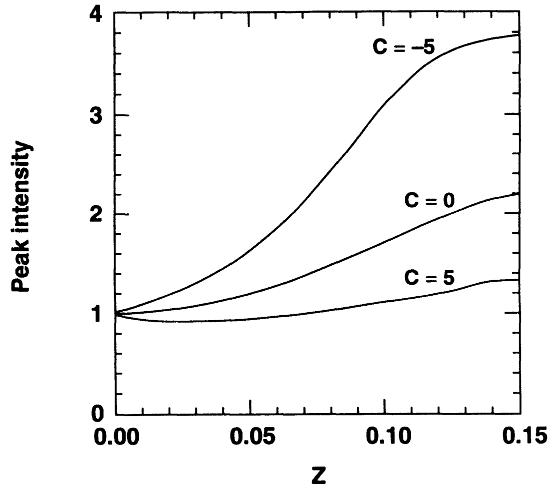

FIG. 2. On-axis intensity of light pulse vs propagation distance for different initial chirp in a medium with normal dispersion for the case of $D=(3+1)$.

In conclusion, self-focusing of a chirped pulse propagating in a nonlinear dispersive medium has been studied analytically and numerically. The quasi-cw moving-foci model is no longer valid when dispersion is important. It has been found that the critical power or wave action for the spatiotemporal self-focusing increases quadratically with increasing chirp in both 2D and 3D. The critical power or wave action depends on the sign of the total chirp parameter. It is shown numerically that the self-

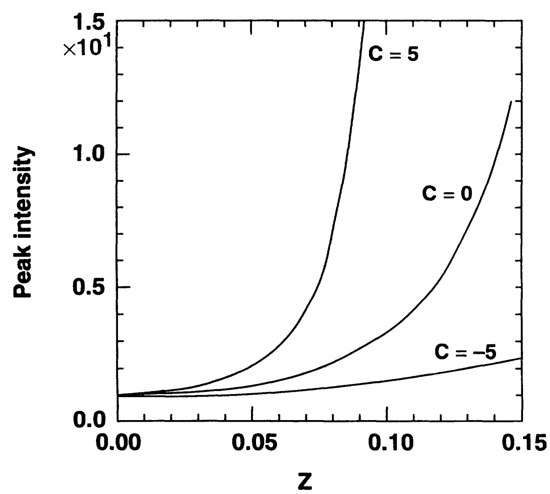

FIG. 3. On-axis intensity of light pulse vs propagation distance for different initial chirp in a medium with anomalous dispersion for the case of $D=(3+1)$.

focusing process can be controlled by changing the chirp parameter. In other words, we can enhance or suppress self-focusing by choosing different chirp parameters. Finally, the chirp introduced initially will be expected to undo itself after the spatiotemporal self-focusing.

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