Space–Time Duality in Optics

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Historical Introduction

- Space-time duality was noted in the 1960s by Tournois and Akhmanov:
  P. Tournois, C. R. Acad. Sci. **258**, 3839–3842 (1964)
  S. A. Akhmanov et al., Sov. Phys. JETP **28**, 748–757 (1969).
- Temporal imaging with a time lens was first discussed in 1989:
  B. H. Kolner and M. Nazarathy, Opt. Lett. **14**, 630–632 (1989)
  B. H. Kolner, IEEE J. Quantum Electron. **130**, 1951–1963 (1994).
- Recent work has focused on applications such as "time microscope" and temporal clocking:
  D. H. Broaddus et al., Opt. Express **18**, 14262–14269 (2010)
  M. Fridman et al., Nature **481**, 62–65 (2012).
- Application of space-time duality to optical signal processing are discussed in a recent review by Alex Gaeta's group:
  R. Salem et al., Adv. Opt. Photon. **5**, 274–317 (2013).

What is Space–Time Duality?

- It results from a mathematical equivalence between paraxial-beam diffraction and dispersive pulse broadening.
- Diffraction in one transverse dimension is governed by

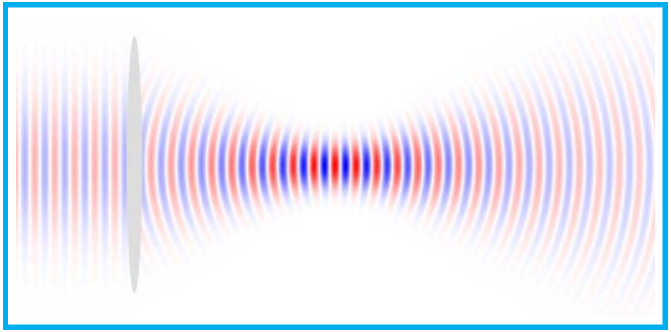
$$\frac{\partial A}{\partial z} + \frac{1}{2ik}\frac{\partial^2 A}{\partial x^2} = 0.$$

- If we neglect higher-order dispersion, pulse evolution is governed by

$$\frac{\partial A}{\partial z} + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} = 0.$$

- Slit-diffraction problem is identical to a pulse propagation problem.
- The only difference is that β_2 can be positive or negative.
- Many results from diffraction theory can be used for pulses.

Concept of a Time Lens



- A lens imposes a quadratic spatial phase shift of the form

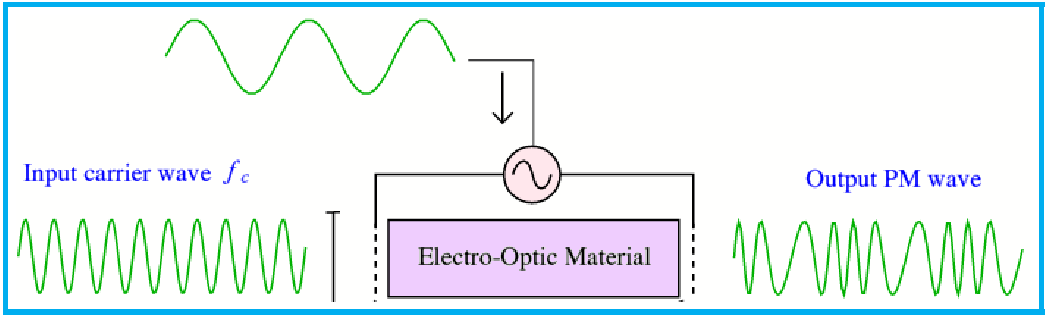
$$A_{\text{out}}(x) = A_{\text{in}}(x) \exp\left(-\frac{ikx^2}{2f}\right).$$

- A time lens must do the same thing in the time domain:

$$A_{\text{out}}(t) = A_{\text{in}}(t) \exp\left(-\frac{it^2}{2D_f}\right).$$

- D_f depends on parameters of the device used to make the time lens.

Phase Modulator as a Time Lens



- A simple way to impose phase shifts is to use an optical phase modulator.
- In the case of sinusoidal modulation at frequency ω_m, we have

$$A_{\rm out}(t) = A_{\rm in}(t)\exp[i\phi_0\cos(\omega_m t)].$$

- If optical pulse is much shorter than one modulation cycle, we can use

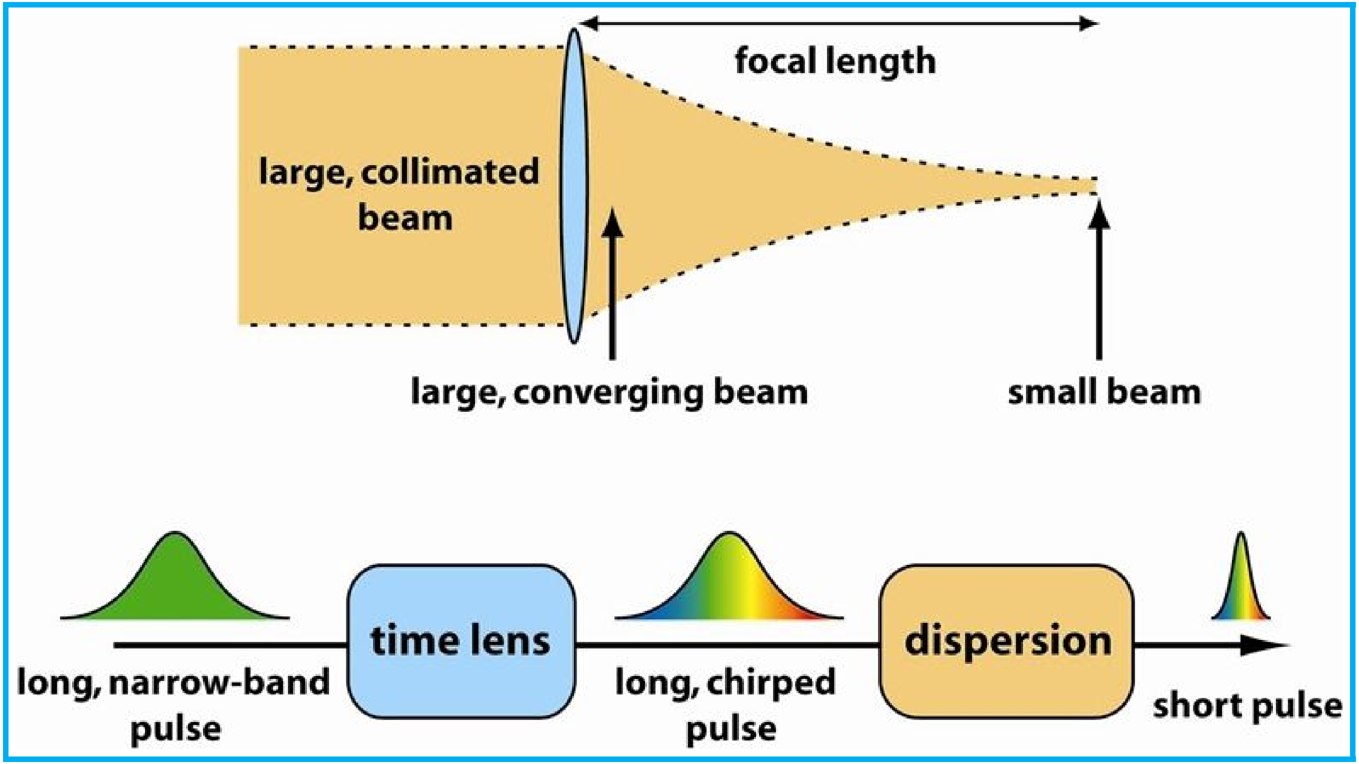
$$A_{\rm out}(t) \approx A_{\rm in}(t)\exp[i\phi_0(1-\omega_m^2 t^2/2)].$$

- In this case $D_f = (\phi_0\omega_m^2)^{-1}$. Its value can be controlled by changing the amplitude and/or frequency of phase modulation.

Techniques for Making a Time Lens

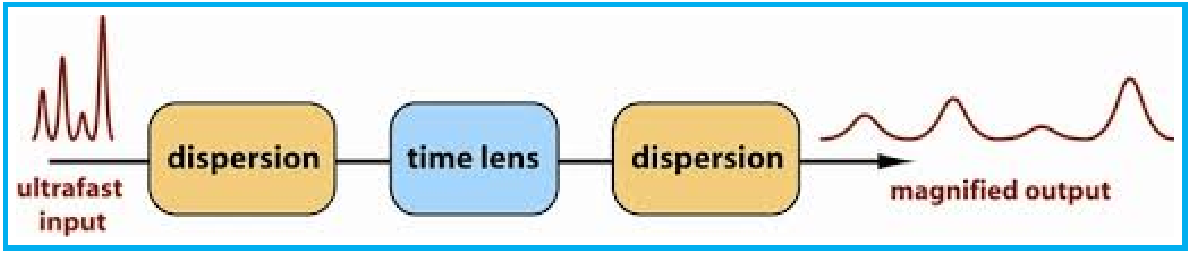
- A quadratic phase shift is equivalent to a linear frequency chirp: $\Delta\omega(t) = -(d\phi/dt) = t/D_f$.
- Any technique that impose a linear chirp on the pulse can be used to make a time lens.
- Many nonlinear techniques can provide a nearly linear frequency chirp.
- Cross-phase modulation by a parabolic pump pulse inside an optical fiber appears to be one possibility.
- Even the use of Gaussian pump pulses in the normal-dispersion region of optical fibers can produce a linear chirp through optical wave breaking.
- Four-wave mixing inside a silicon waveguide, or an optical fiber, has been used in several recent experiments.

Focusing by a Time Lens



R. Salem et al., Adv. Opt. Photon. **5**, 274–317 (2013)

Temporal Focusing and Imaging

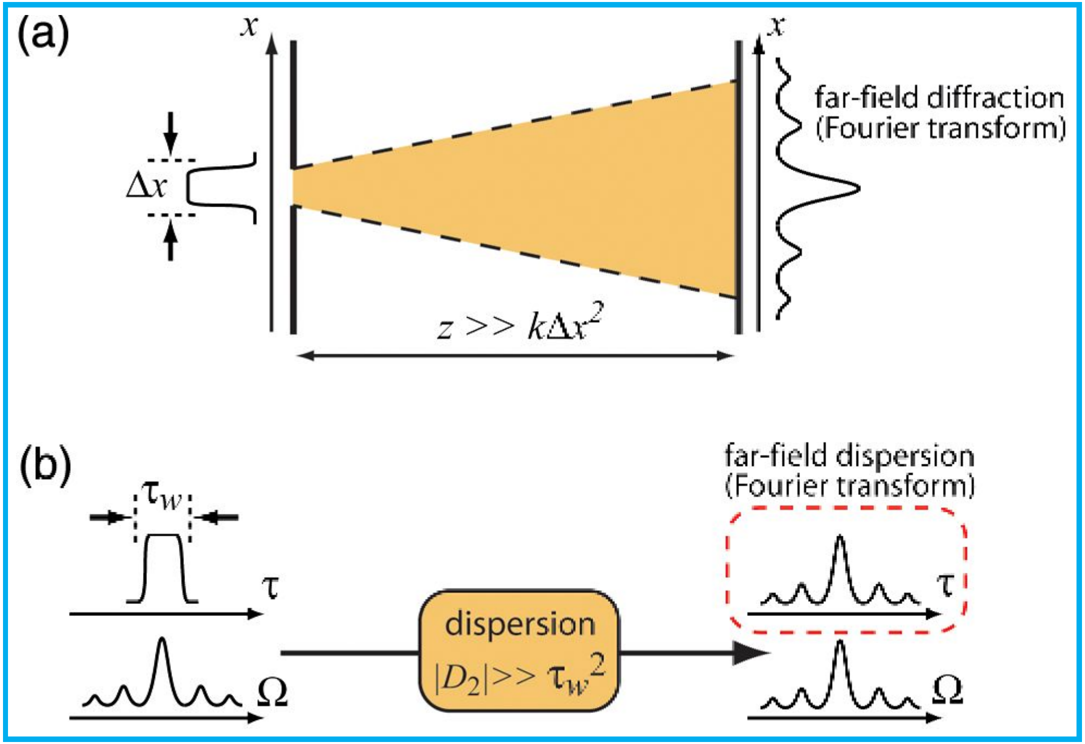


- A time lens, followed by a dispersive medium of suitable length, can compress optical pulses through temporal focusing.
- A temporal imaging system requires two dispersive sections.
- It can be used to make a time microscope that magnifies optical pulses.
- The imaging condition for a time lens is found to be

$$\frac{1}{D_1}+\frac{1}{D_2}=\frac{1}{D_f}, \qquad D_n=\beta_{2n}L_n, \quad D_f=\frac{1}{\phi_0\omega_m^2}.$$

- D_f is called the focal GDD (Group Delay Dispersion) of a time lens.

Temporal Fourier Transform



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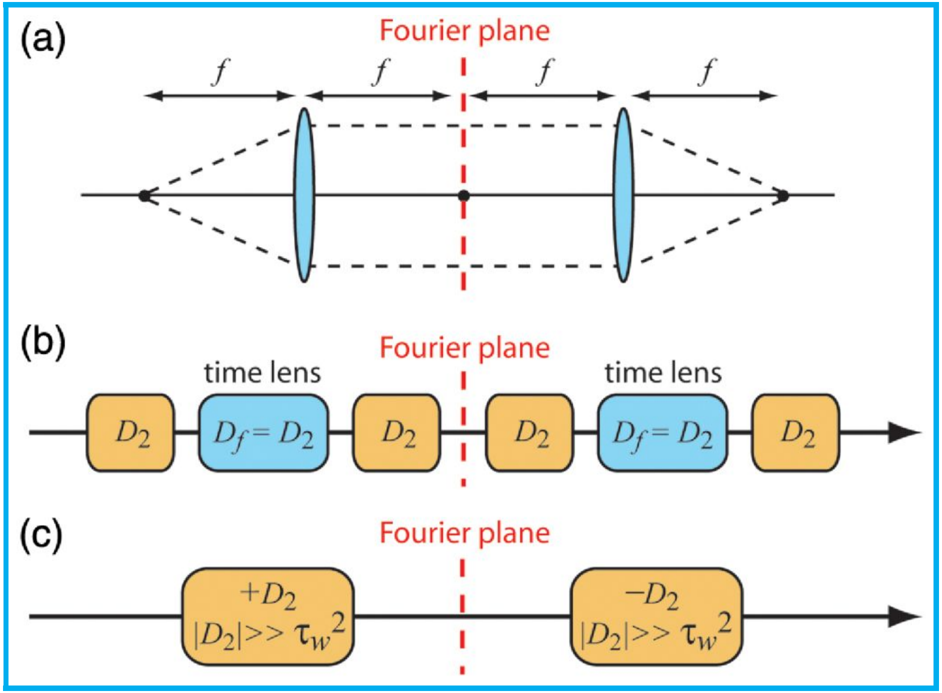
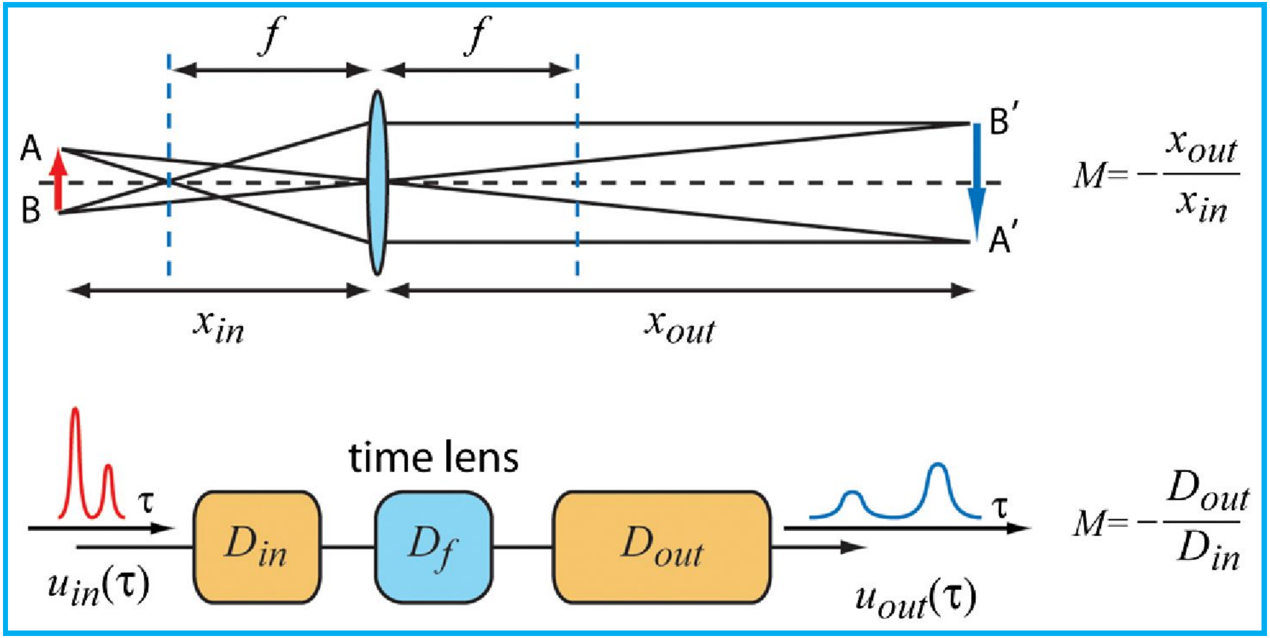
Temporal 4-f Processor



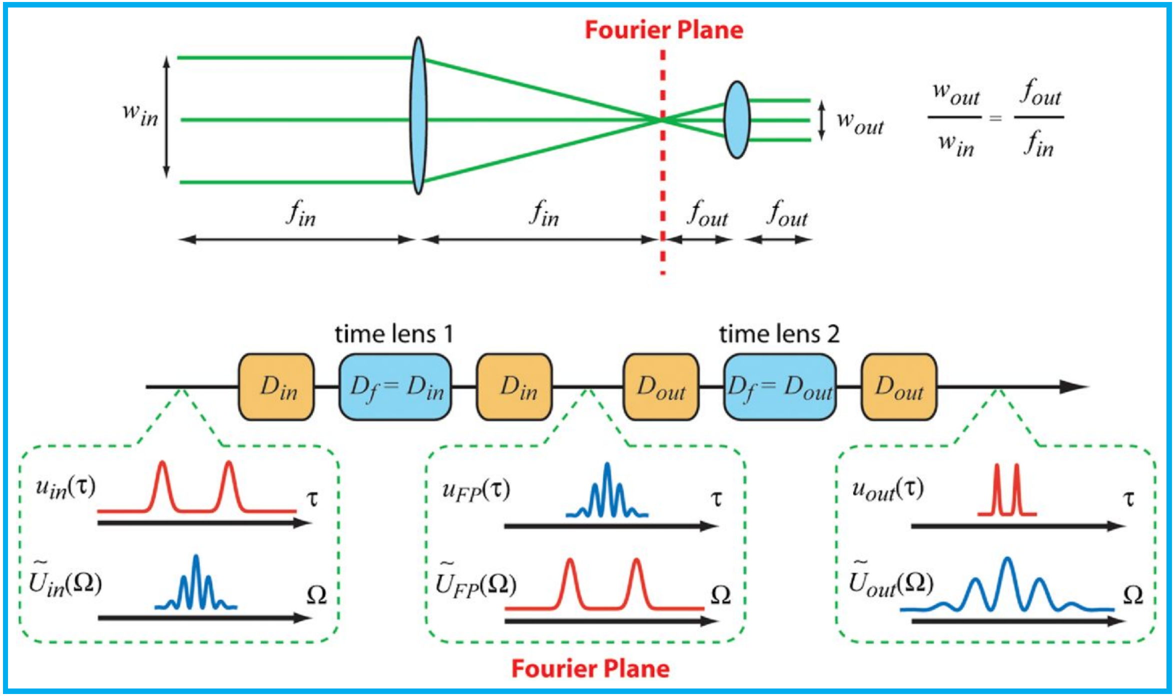
R. Salem et al., Adv. Opt. Photon. **5**, 274–317 (2013)

Temporal Microscope



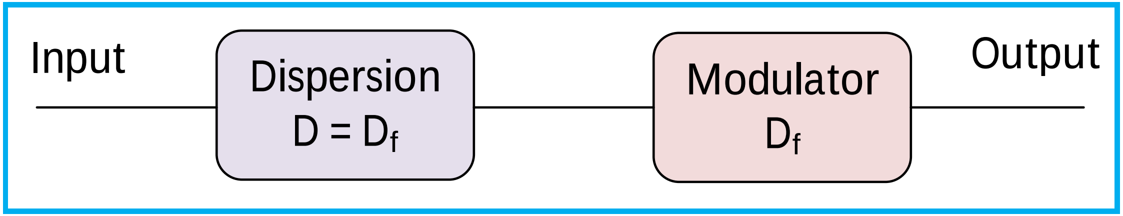
R. Salem et al., Adv. Opt. Photon. **5**, 274–317 (2013)

Temporal Telescope



R. Salem et al., Adv. Opt. Photon. **5**, 274–317 (2013)

Modulator-Induced Spectral Changes



- For this configuration, the input and output spectra are related as

$$A_o(\omega) = \frac{1}{2\pi} \iint A_i(\omega') \exp(i\beta_2 L\omega'^2/2) e^{i\phi_m(t)} e^{i(\omega-\omega')t} d\omega' \, dt.$$

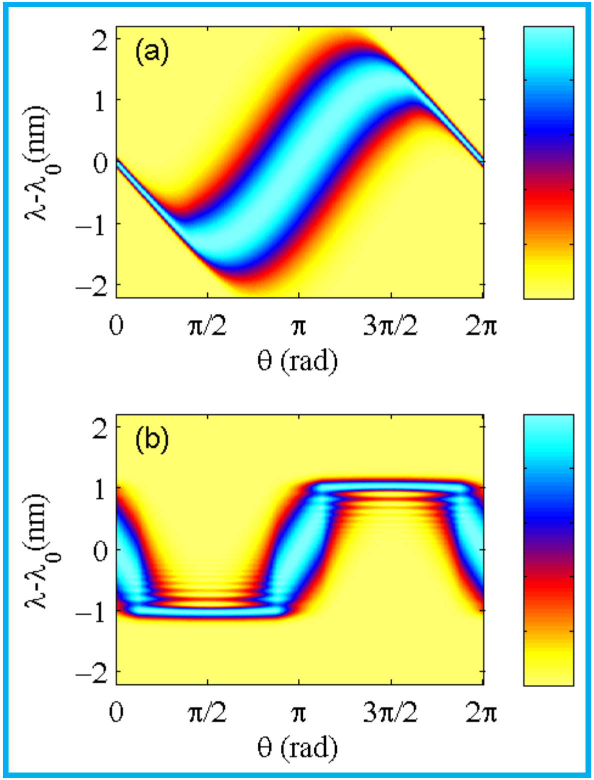
- When peak of the pulse does not coincide with the modulation peak,

$$\phi_m(t) = \phi_0 \cos(\omega_m t - \theta) \approx \phi_0[\cos\theta - \omega_m t \sin\theta - \omega_m^2 t^2 \cos\theta/2]$$

- The linear term produces a spectral shift; it vanishes for $\theta = m\pi$.
- The quadratic term chirps the pulse; it vanishes for $\theta = m\pi/2$.

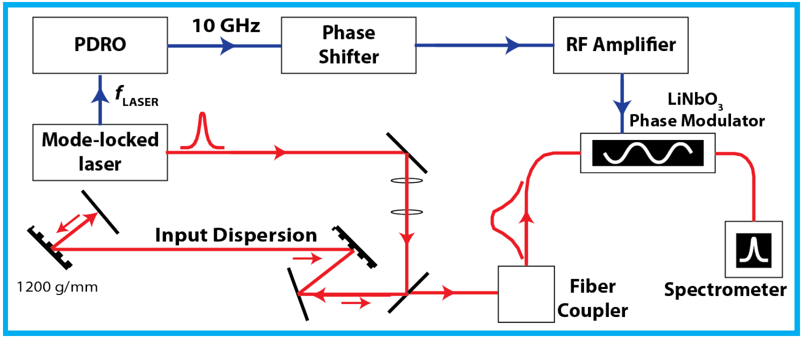
Numerical Results

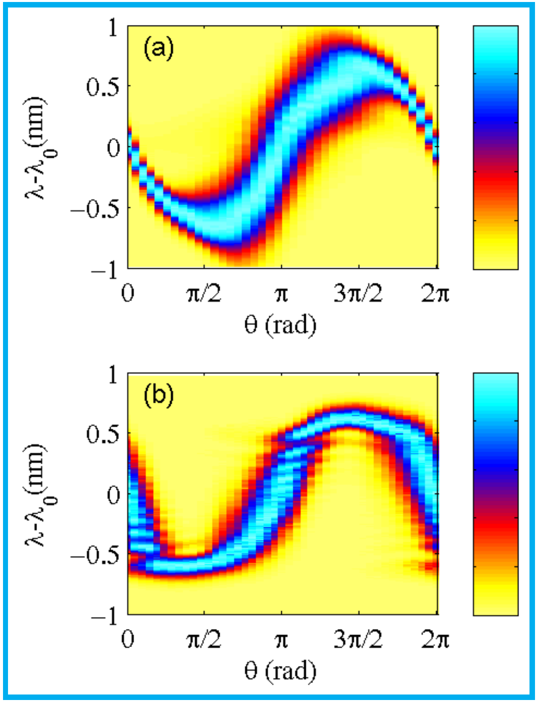
- Output spectra versus θ:
  (a) 1.5 ps Gaussian pulses.
  (b) 20 ps Gaussian pulses.
- Modulation frequency 10 GHz;
  $\phi_0 = 30$ rad ($D_f = 8.44$ ps^2);
  Time aperture $1/\omega_m = 16$ ps.
- Spectrum is narrowest for $\theta = 0$; it shifts and broadens as θ increases. Reverse spectral changes occur after $\theta = \pi$.
- Considerable distortions occur for pulses broader than the aperture of time lens.



Experimental Results

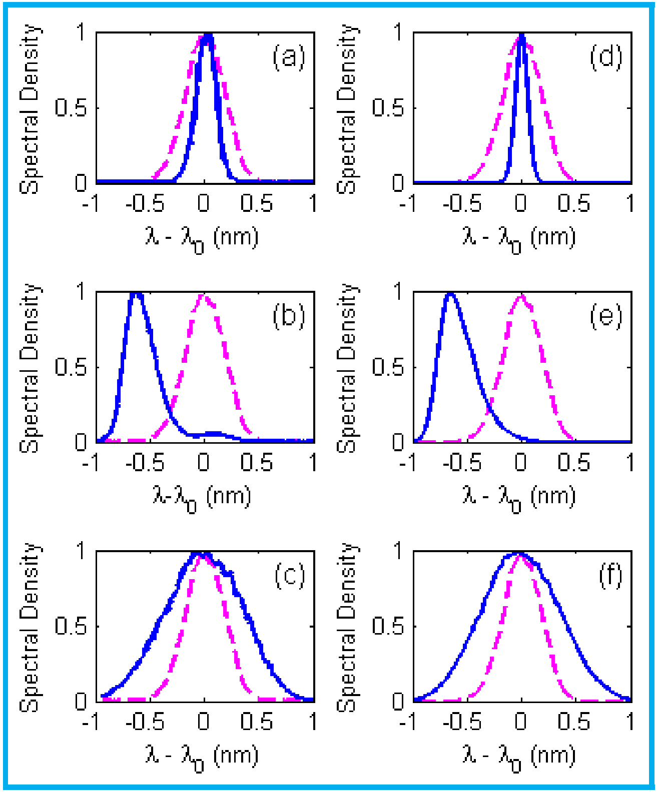
Experimental setup





Plansinis et al., JOSA B **32** (August 2015)

Modulator-Induced Spectral Changes

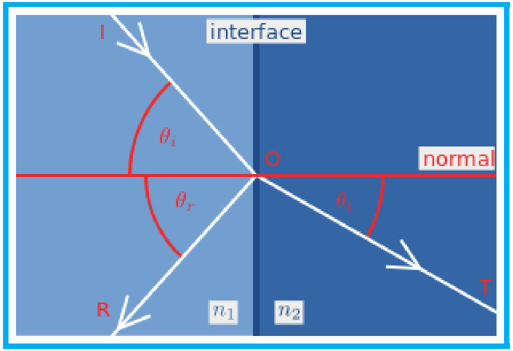


- Experiment on left
- Theory on right
- Blue: Output spectrum
- Purple: Input spectrum
- Top: $\theta = 0$
- Middle: $\theta = \pi/2$
- Bottom: $\theta = \pi$

Plansinis et al., JOSA B **32** (August 2015)

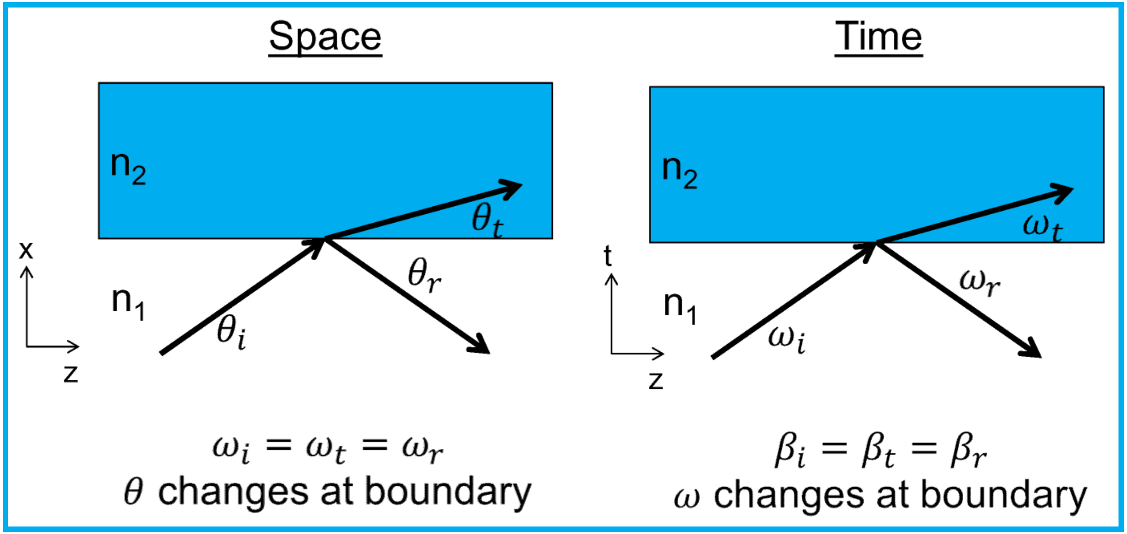
Temporal Reflection and Refraction

- Reflection and refraction of optical beams at a spatial boundary are well-known phenomena.
- What is the temporal analog of these two optical phenomena?



- What happens when an optical pulse arrives at a temporal boundary across which refractive index changes suddenly?
- At a spatial boundary, energy is preserved but momentum can change.
- At a temporal boundary, momentum is preserved but frequency can change.
- A change in angle at a spatial interface translates into a change in the frequency of incident light.

Space–Time Duality



- Comparison of reflection and refraction in space and time
- Frequency conserved but wave vector changes in the spatial case.
- Wave vector conserved but frequency changes in the temporal case.

Simple Model of Pulse Propagation

- Let us assume that an optical pulse is propagating inside a waveguide with the dispersion relation $\beta(\omega)$
- Temporal discontinuity at $t = T_B$ is incorporated by using

$$\beta(\omega) = \beta_0 + \Delta\beta_1(\omega - \omega_0) + \frac{\beta_2}{2}(\omega - \omega_0)^2 + \beta_B H(t - T_B).$$

- $\Delta\beta_1 = \beta_1 - \beta_{1B}$ is pulse's relative speed relative to the temporal boundary located at $t = T_B$.
- $\beta_B = k_0 \Delta n$, if refractive index changes by Δn for $t > T_B$; $H(x)$ is the Heaviside function.
- This dispersion relation can be used to investigate changes in pulse's shape and spectrum occurring after the pulse arrives at the boundary.

Pulse Propagation

- Slowly varying envelope of the pulse satisfies

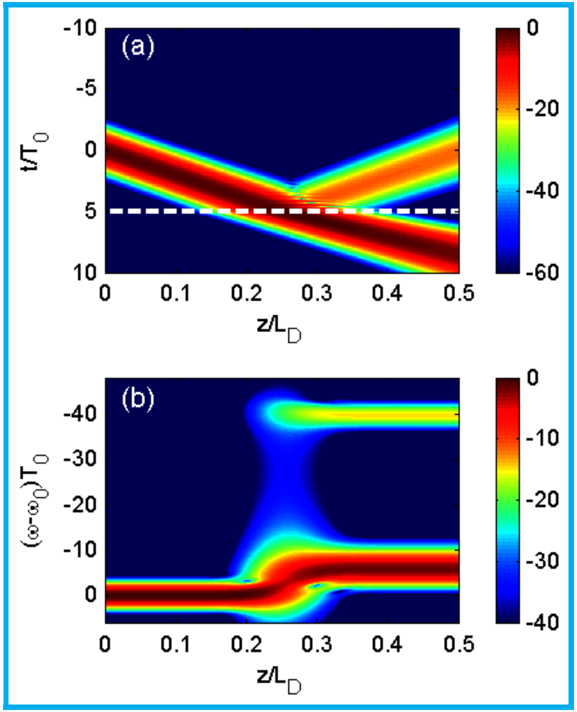
$$\frac{\partial A}{\partial z} + \Delta\beta_1 \frac{\partial A}{\partial t} + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} = i\beta_B H(t - T_B)A.$$

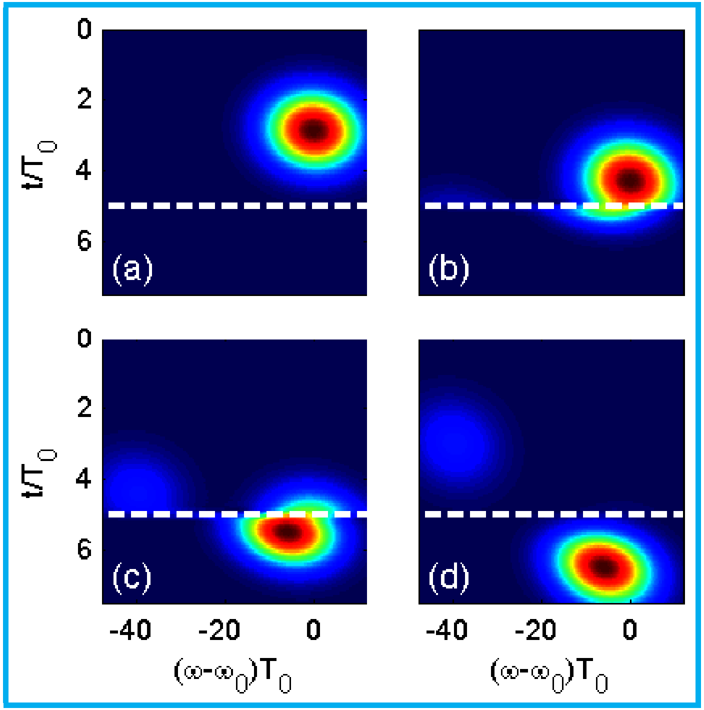
- Using $\tau = t/T_0$ and $\xi = z/L_D$ ($L_D = T_0^2/|\beta_2|$), the normalized form becomes

$$\frac{\partial A}{\partial \xi} + d\frac{\partial A}{\partial \tau} + \frac{ib_2}{2}\frac{\partial^2 A}{\partial t^2} = i\beta_B L_D H(\tau - T_B/T_0)A.$$

- Numerical results obtained for Gaussian pulses with the temporal boundary at $T_B = 5T_0$ using $\beta_B L_D = 100$ and $d = \Delta\beta_1 L_D/T_0 = 20$.
- Pulses reaches the boundary at a distance of $z = L_D/4$.
- Temporal evolution of the pulse shows a clear evidence of both the reflection and refraction at the boundary.

Temporal Reflection and Refraction

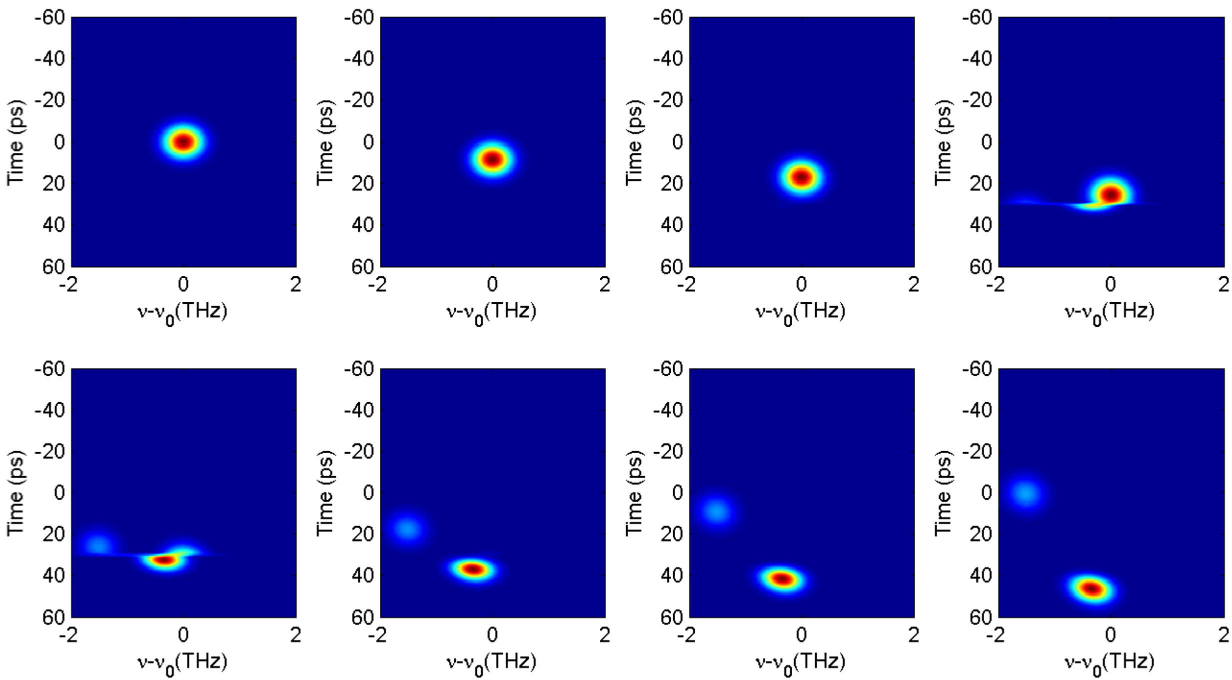




Plansinis et al., PRL (submitted).

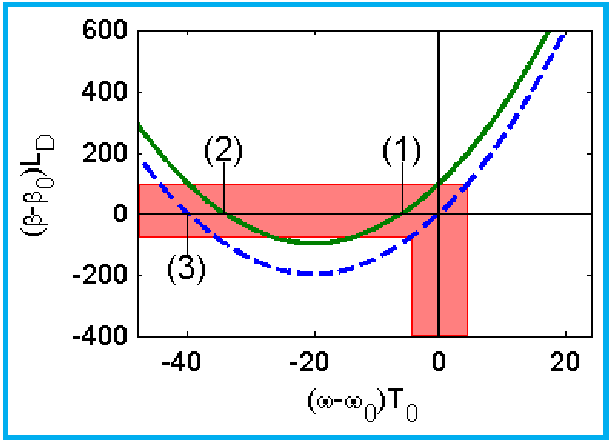
Spectrograms for 8-ps Gaussian pulses

Gaussian Input Pulse, T_0 = 5ps, T_B = 30 ps, $\Delta\beta_1$ = 300 ps/km, β_2 = 63 ps^2/km



Momentum Conservation

- Momentum conservation explains all results:
  - ⋆ Blue curve for $t < T_B$
  - ⋆ Green curve for $t > T_B$
  - ⋆ Red region: pulse spectrum
- Possible solutions marked



- Reflection corresponds to solution (3) on the blue curve.
- Refraction corresponds to solution (1) on the blue curve.
- Solution (2) is not physical since its slope is opposite to that of (1).
- Both reflection and refraction manifest as red-shifted pulses; blue shifts occur if β_2 or Δn is negative.

Spectral Shift of Reflected Pulse

- Momentum is related to $\beta(\omega)$ given by

$$\beta(\omega) = \beta_0 + \Delta\beta_1(\omega - \omega_0) + \frac{\beta_2}{2}(\omega - \omega_0)^2 + \beta_B H(t - T_B).$$

- For $\omega = \omega_0$, we need to maintain $\beta = \beta_0$, resulting in

$$\Delta\beta_1(\omega - \omega_0) + \frac{\beta_2}{2}(\omega - \omega_0)^2 + \beta_B H(t - T_B) = 0.$$

- Reflected pulse is confined to the region $t < T_B$. The only solution is

$$\omega_r = \omega_0 - 2(\Delta\beta_1/\beta_2).$$

- The sign and magnitude of the spectral shift of the reflected pulse depend on values of $\Delta\beta_1$ and β_2.

Spectral Shift of Refracted Pulse

- We need to satisfy the phase-matching condition:

$$\Delta\beta_1(\omega-\omega_0)+\frac{\beta_2}{2}(\omega-\omega_0)^2+\beta_B H(t-T_B)=0.$$

- Refracted pulse propagates to the region $t>T_B$ where $H(t-T_B)=1$. The quadratic equation has the solutions

$$\omega_t=\omega_0+\frac{\Delta\beta_1}{\beta_2}\left(-1\pm\sqrt{1-\frac{2\beta_B\beta_2}{(\Delta\beta_1)^2}}\right).$$

- Only $+$ sign corresponds to the physical solution. In the limit $\Delta\beta_1\gg\sqrt{\beta_B\beta_2}$, it can be approximated as $\omega_t=\omega_0-(\beta_B/\Delta\beta_1)$.
- The sign and magnitude of the spectral shift of the refracted pulse depend on values of $\Delta\beta_1$, β_2, and β_B.

Total Internal Reflection

- Total internal reflection (TIR) can occur in the spatial case when an optical beam enters from a high-index medium to low-index medium ($\Delta n < 0$).
- The condition for its occurrence is obtained from Snel's law, $n_1 \sin\theta_1 = n_2 \sin\theta_2$, by setting $\theta_2 = 90^\circ$.
- We don't have such a simple law for temporal TIR.
- One way to find the condition for TIR is to see when the spectral shift of the refracted pulse becomes unphysical:

$$\omega_t = \omega_0 + \frac{\Delta\beta_1}{\beta_2}\left(\sqrt{1 - \frac{2\beta_B\beta_2}{(\Delta\beta_1)^2}} - 1\right).$$

- This condition is clearly $2\beta_B\beta_2 > (\Delta\beta_1)^2$. TIR occurs only if β_B and β_2 have the same signs.

Total Internal Reflection

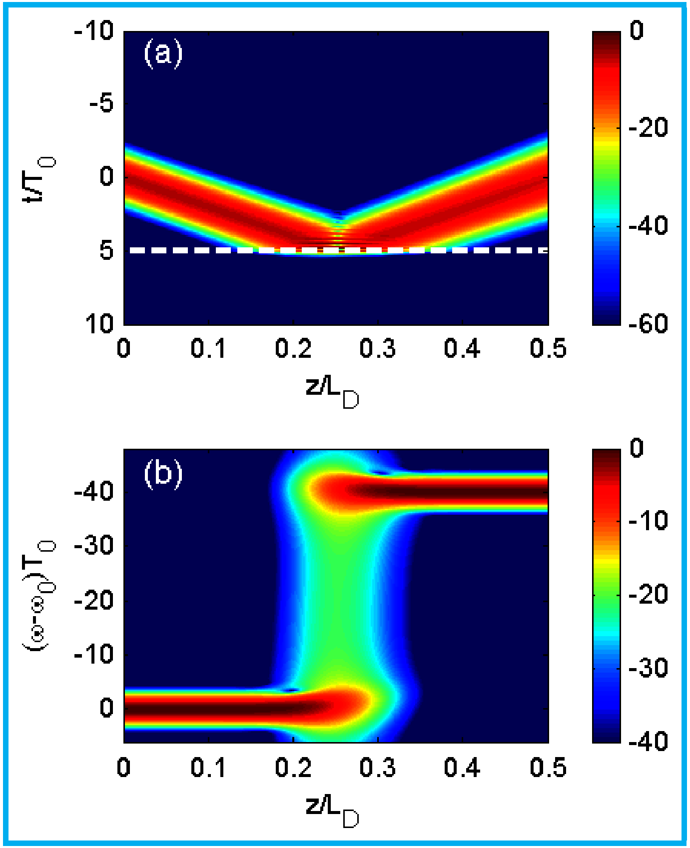
- Temporal TIR is not restricted to the situation $\Delta n < 0$.
- Using $\beta_B = k_0 \Delta n$, we can write the condition for TIR as

$$\beta_2 \Delta n > (\Delta\beta_1)^2/2k_0.$$

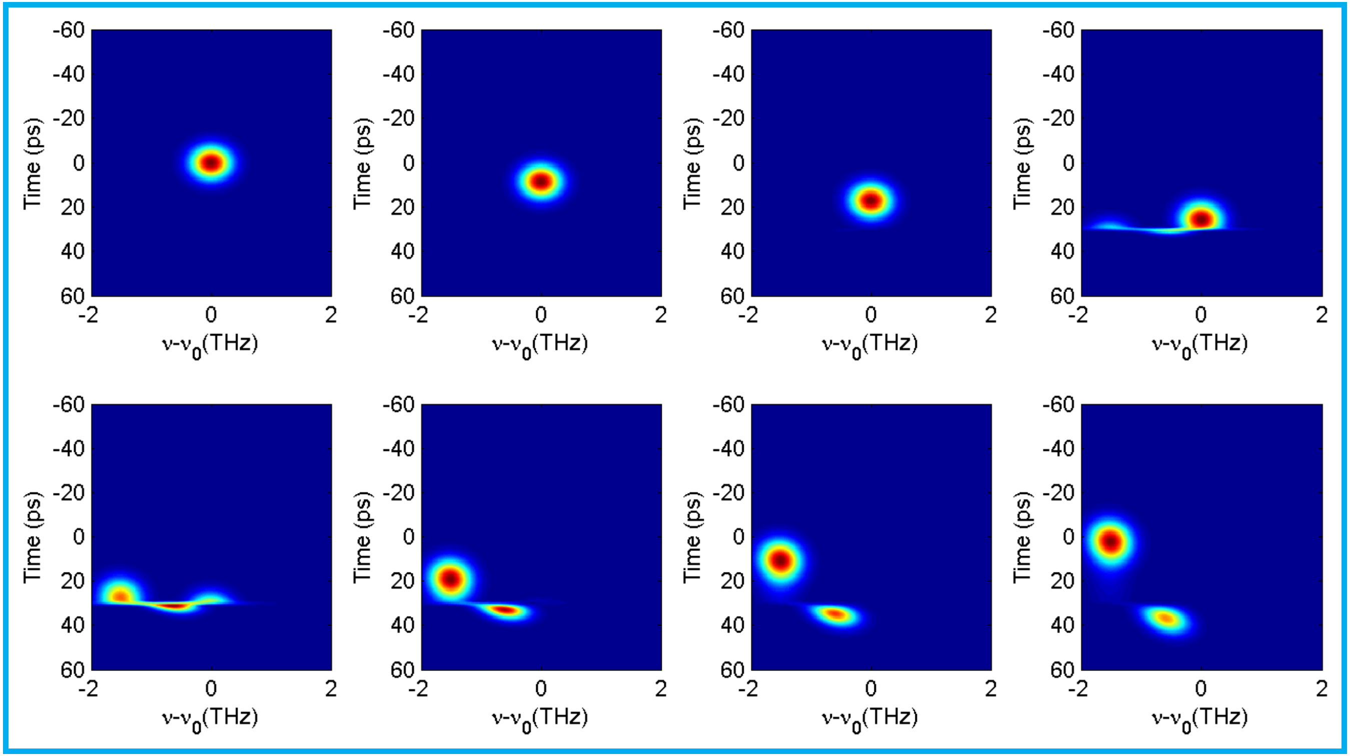
- When $\Delta n > 0$, the pulse needs to propagate in the normal-dispersion region.
- In contrast, the pulse must propagate in the anomalous-dispersion region when $\Delta n < 0$.
- This freedom is a consequence of the fact that dispersion term can be positive or negative whereas the diffraction term has only one sign.
- The requirement $\Delta n > (\Delta\beta_1)^2/(2k_0\beta_2)$ can be satisfied in practice even for $\Delta n \sim 10^{-4}$.

TIR of Gaussian Pulses

- Evolution of a Gaussian pulse
  (a) Temporal evolution
  (b) Spectral evolution
- Temporal boundary located at $T_B = 5T_0$.
- Index change was large enough ($\beta_B L_D = 320$) to satisfy the TIR condition.
- Entire pulse energy reflected at the temporal boundary
- Spectrum shifted toward the red side during TIR if $\Delta n > 0$.



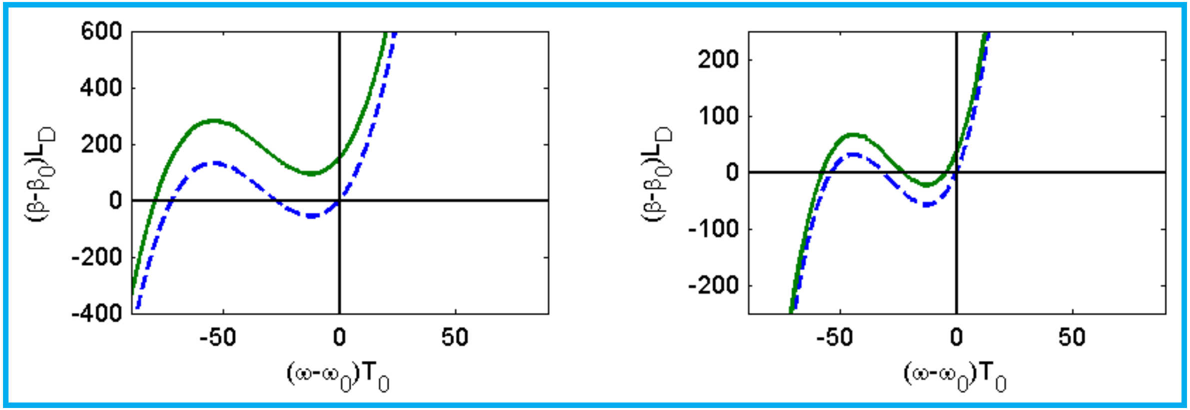
Spectrograms for 8-ps Gaussian pulses



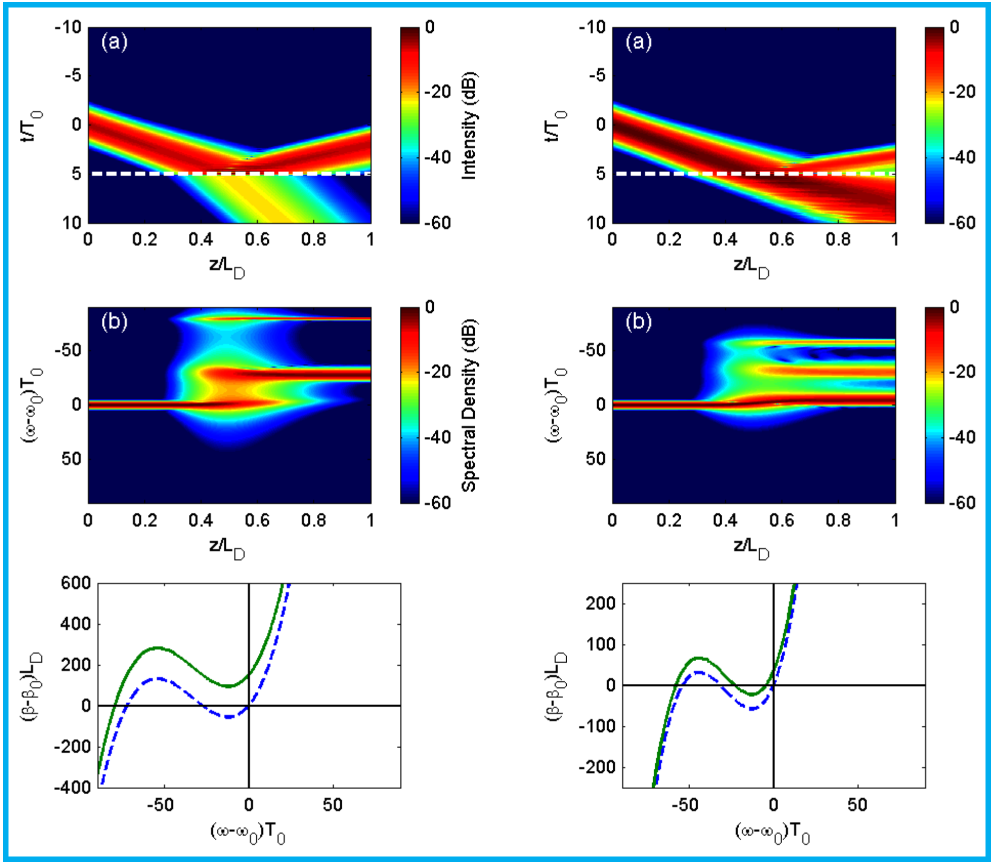
Effects of Third-Order Dispersion

- Third-order dispersion (TOD) leads to distortions in temporal imaging.
- In the case of temporal reflection, TOD can lead to two reflected pulses.
- The phase-matching condition now becomes a cubic polynomial:

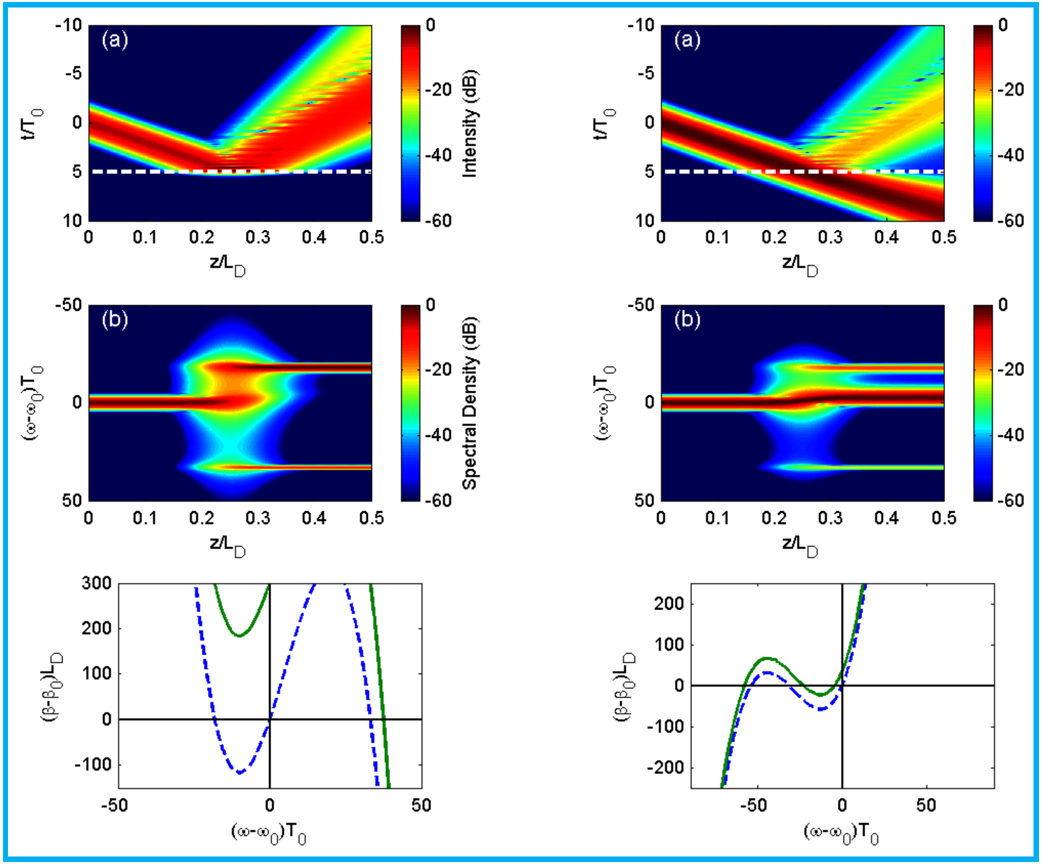
$$\Delta\beta_1(\omega-\omega_0)+\frac{\beta_2}{2}(\omega-\omega_0)^2+\frac{\beta_3}{6}(\omega-\omega_0)^3+\beta_B H(t-T_B)=0.$$



Double Refraction

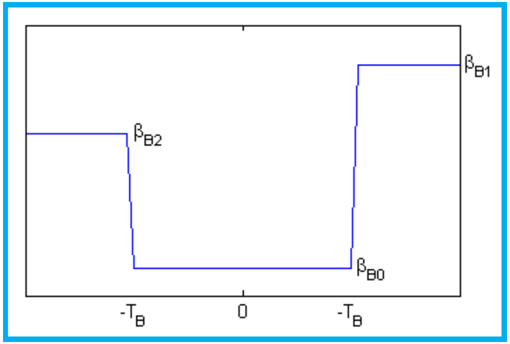


Double Reflection



Temporal Waveguides

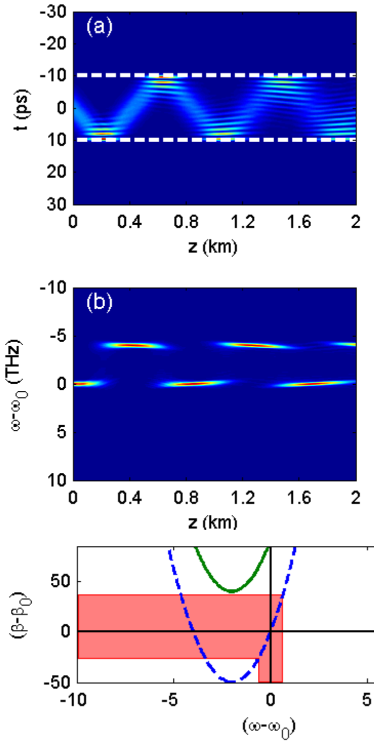
- TIR can be used to make temporal waveguides that trap optical pulses.
- Two temporal boundaries are needed.
- Central region can have lower or higher refractive index.



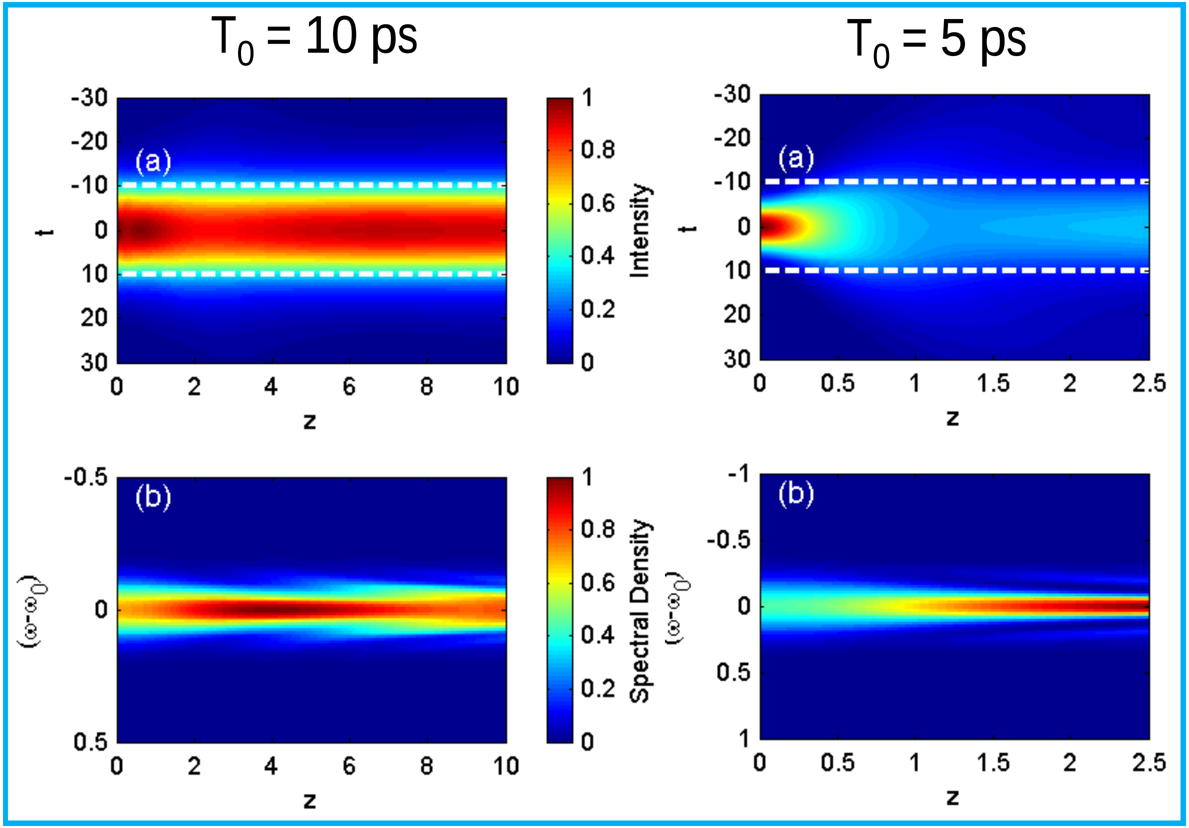
- Modes of a temporal waveguide are similar to those of spatial waveguides.
- A pulse can remain trapped inside the waveguide if it undergoes TIR at both temporal boundaries.
- This technique has the potential of creating pulses that remain confined to a fixed time window.

Pulse Trapping

- A 8-ps Gaussian pulse ($T_0 = 5$) trapped inside a 20-ps wide waveguide:
  $\Delta\beta_1 = 50$ ps/km, $\beta_2 = 25$ ps^2/km,
  $\beta_B = 90$ km^{-1}
- Pulse undergoes TIR and its spectrum shifts after each reflection.
- Pulse broadening eventually creates distortions and the pulse excites several modes of a multimode waveguide ($V = 26.8$).
- This approach can work only when $T_0 \ll T_B$ and $z \ll L_D$. It is better to design a single-mode temporal waveguide.



Single-Mode Waveguide



Conclusions

- Space–time duality is a simple concept with many applications.
- It can be used to stretch and compress optical pulses.
- It can be used for temporal imaging and making time microscopes.
- A phase modulator acting as a time lens can lead to spectral changes.
- Temporal equivalent of reflection and refraction occurs for optical pulses at a temporal boundary.
- In the temporal case, frequency plays the role of angles.
- We have identified conditions under which a pulse undergoes total internal reflection and studied the effects of higher-order dispersion.
- The TIR phenomenon can be used to trap pulses within a fixed time window (temporal wave-guiding).